Introduction to S-matrix Bootstrap

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Outline

1. Introduction: main idea and axioms
2. The methods: primal and dual approaches
3. Applications: 2D maps, non-invertible symmetries, pions
4. Final Remarks

Introduction

The S-matrix Program

- Motivation: describe strong interactions

- Idea of 1960's program: [e.g. Eden et al 1966, Chew 1966]

  Find a unique **non-perturbative S-matrix** by imposing **consistency conditions** coming from general principles ← Bootstrap philosophy

- Modern S-matrix Bootstrap: constrain space of consistent QFTs *(weak and strongly coupled)* [Paulos, Penedones, Toledo, van Rees, Vieira '16]

  → bound physical parameters

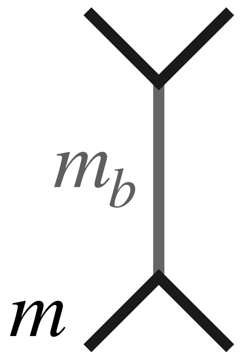
  → chart the landscape of QFTs

The Axioms (ACU)

Analyticity (causality)

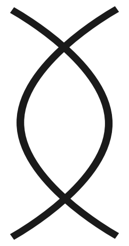
Amplitudes are boundary values of analytic functions of complexified kinematic invariants

Landau analyticity: only singularities from on-shell processes



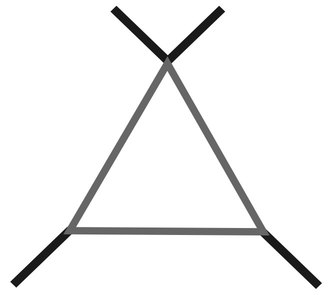
$\sim \frac{g^2}{s - m_b^2}$

Bound states



$s \geq 4m^2$

Normal thresholds



extra singularities @ s_*

Anomalous thresholds

* Only partial proven results for theories with mass gap

Crossing

Amplitudes in different kinematic channels are boundary values of the <u>same</u> analytic function

Relates process in which we exchange in ↔ out particles

* Beyond 4 particles → inclusive observables [Caron-Huot, Giroux, Hannesdottir, Mizera '23]

Unitarity

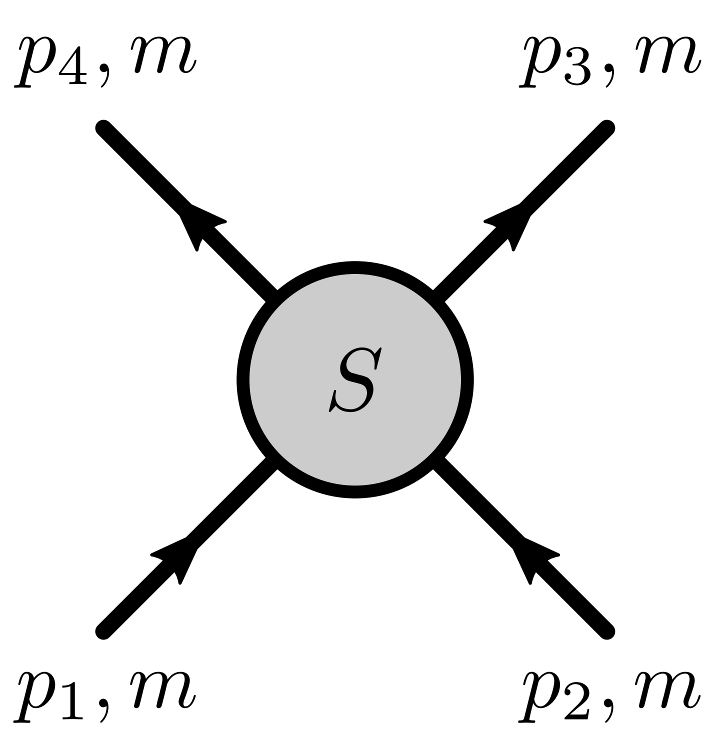
S-matrix is the evolution operator relating in and out basis of states. Conservation of probabilities

$$\mathbb{S}^\dagger \mathbb{S} = \mathbb{I}$$

The standard amplitude: lightest 2→2

Most bootstrap studies focus on 2 → 2 scattering of the lightest particle in the theory *(no anomalous thresholds)*

$\mathbb{S} = \mathbb{I} + i\mathbb{T}$ $\quad \langle p_3, p_4 | \mathbb{T} | p_1, p_2 \rangle = (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)\, T(s,t)$



$s = (p_1+p_2)^2,\ \ t = (p_1 - p_4)^2,\ \ u = (p_1 - p_3)^2$

$s + t + u = 4m^2$

Crossing

$$T(s,t) = T(t,s) = T(u,t)$$

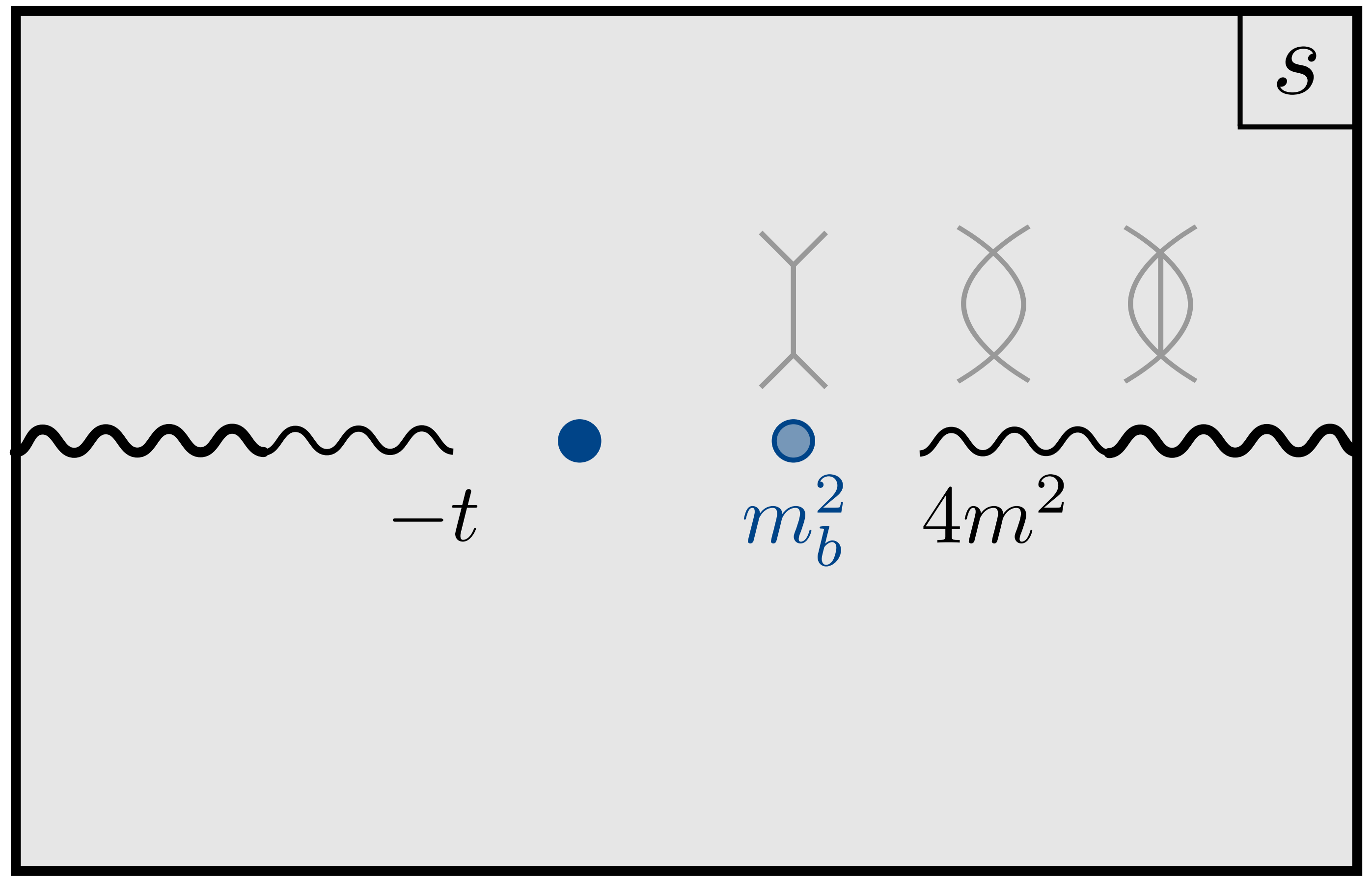
Unitarity

$$\mathbb{S}^\dagger \mathbb{S} = \mathbb{I} \implies |S_{2\to 2}|^2 + \sum_{n>2} |S_{2\to n}|^2 = 1$$

Partial waves $S_l(s) = 1 + i\frac{\pi}{4}\sqrt{\frac{s-4m^2}{s}} \int_{-1}^{1} d\cos\theta\, P_l(\cos\theta)\, T(s,t)$

$$|S_l(s)|^2 \leq 1,\ \ s > 4m^2$$

Analyticity



Dispersion relation (fixed t)

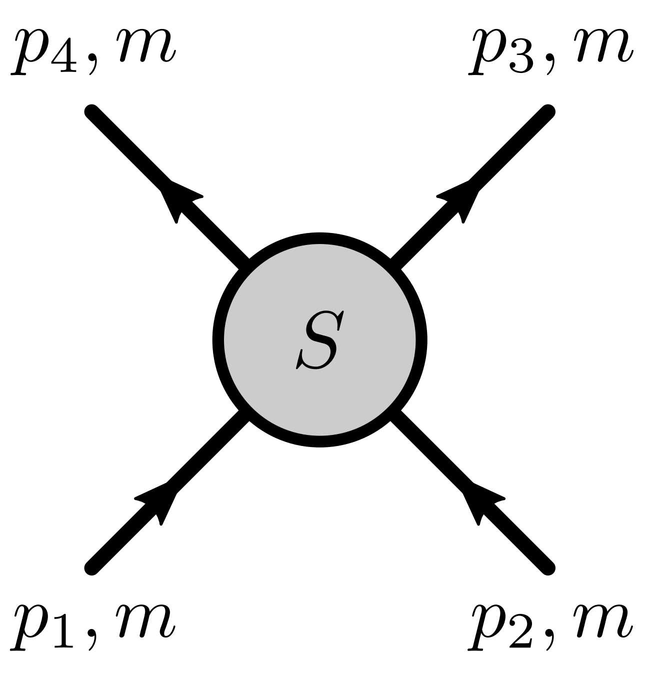
$$T(s,t) = \text{poles} + \int_{4m^2}^{\infty} \frac{ds'}{\pi} \frac{T_s(s',t)}{s'-s} + \int_{4m^2}^{\infty} \frac{dt'}{\pi} \frac{T_u(u',t)}{u'-u}$$

(ignoring 2 subtractions from Froissart bound $T(s,t) \lesssim s \ln^2 s$)

The Methods

Warm-up: single particle in 2D

A simple bootstrap problem solved with complex analysis.



Lorentz, 2D

$$S(p_1, p_2, p_3, p_4) = S(s)$$

$$s + t + \not{u} = 4m^2$$

Crossing

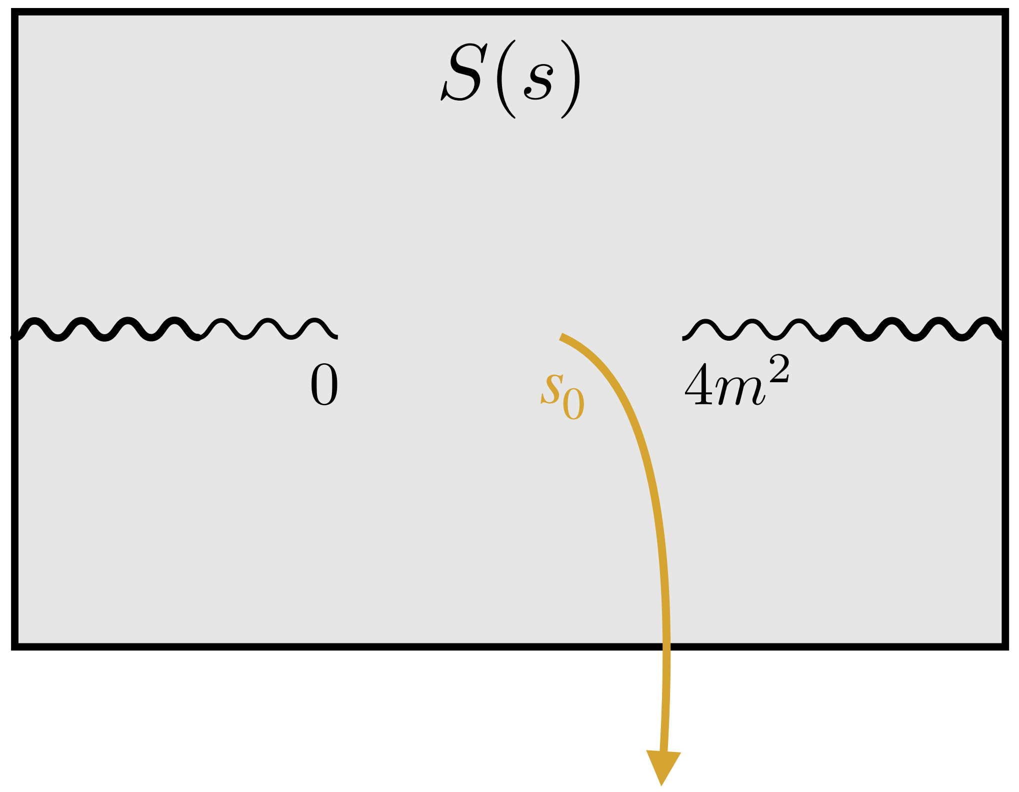
$$S(s) = S(t = 4m^2 - s)$$

Unitarity

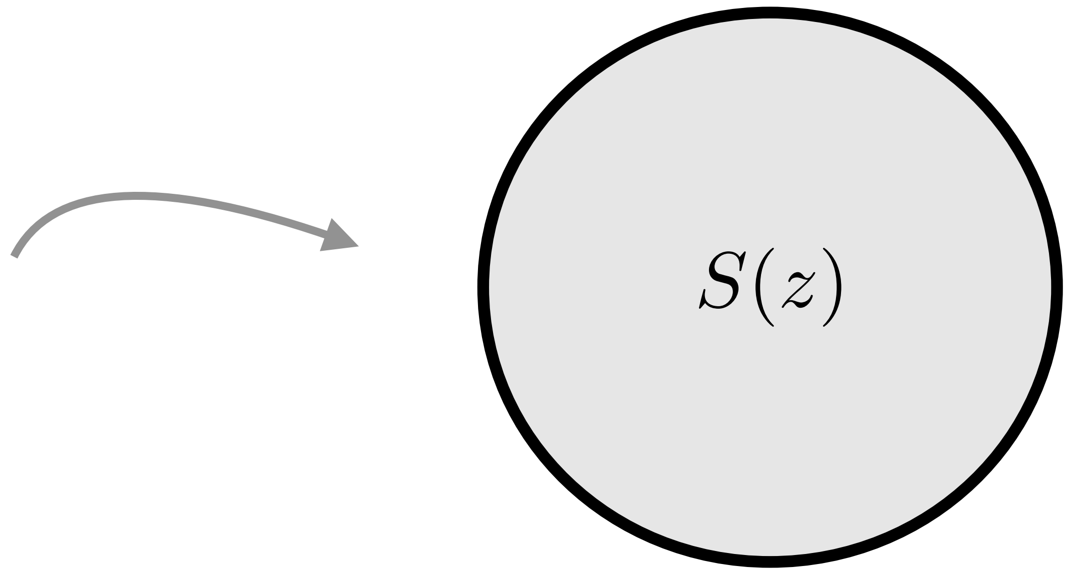
$$|S(s + i0)|^2 \leq 1, \qquad s > 4m^2$$

Analyticity

Suppose there are no bound states



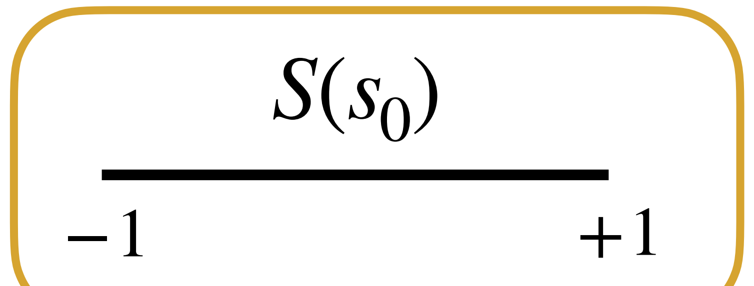
$$z(s) = \frac{\sqrt{s}\sqrt{4m^2 - s} - \sqrt{s_0}\sqrt{4m^2 - s_0}}{\sqrt{s}\sqrt{4m^2 - s} + \sqrt{s_0}\sqrt{4m^2 - s_0}}$$



What is max/min $S(s_0)$?

Maximum modulus principle: $|S(z)|$ has its maximum at the boundary (or is a constant).

$\rightarrow$ $|S(s_0)| \leq 1$ and saturated by **free** scattering $S = \pm 1$



Primal approach

Constructing ACU amplitudes numerically

Bootstrap recipe

Given a spectrum $\{m, m_b, \ldots\}$

What is the space of allowed amplitudes?

1. Write ansatz that trivializes **A**+**C**
2. Impose **U** numerically for $s_j > 4m^2$
3. Bound parameters, i.e. max functionals $\mathcal{F}$

4D analogue

1. $T(s,t) = \sum_{a,b,c=0}^{N_{max}} \alpha_{(abc)} \rho_s^a \rho_t^b \rho_u^c$
2. $|S_l(s)|^2 \leq 1, \;\; s_j > 4m^2, \;\; l \leq L_{max}$
3. $(32\pi)^{-1}\, T\left(4m^2/3, 4m^2/3\right)$

   -6.6 ——— 2.66

[Paulos, Penedones, Toledo, van Rees, Vieira '17]

2D example

$2 \rightarrow 2$, no bound states $\{m\}$

1. "Rho" ansatz (or dispersion relations)

$$S(s) = \sum_{n=0}^{N_{max}} \alpha_n \left[\rho_s^n + \rho_t^n\right] \,, \qquad \rho_s = \frac{\sqrt{4m^2 - s_0} - \sqrt{4m^2 - s}}{\sqrt{4m^2 - s_0} + \sqrt{4m^2 - s}}$$

2. $|S(s_j)|^2 \leq 1, \;\; s_j > 4m^2, \;\; j \leq N_{grid}$
3. Max/min $\mathcal{F}[S(s)] = S(2m^2)$

$S(2m^2)$

-1 ——— $+1$

But… bounds depend on truncation parameters N_{max} , L_{max} …

Dual approach

Deriving rigorous bounds from dual functionals [2D: LC, He, Kruczenski, Vieira '19; Guerrieri, Homrich, Vieira '20; Elias, Miro, Guerrieri '21; 4D: He, Kruczenski '21; Guerrieri, Sever '21]

$$\text{Max } \mathscr{F}[S] \leq \text{Min } \mathscr{F}_d[K]$$

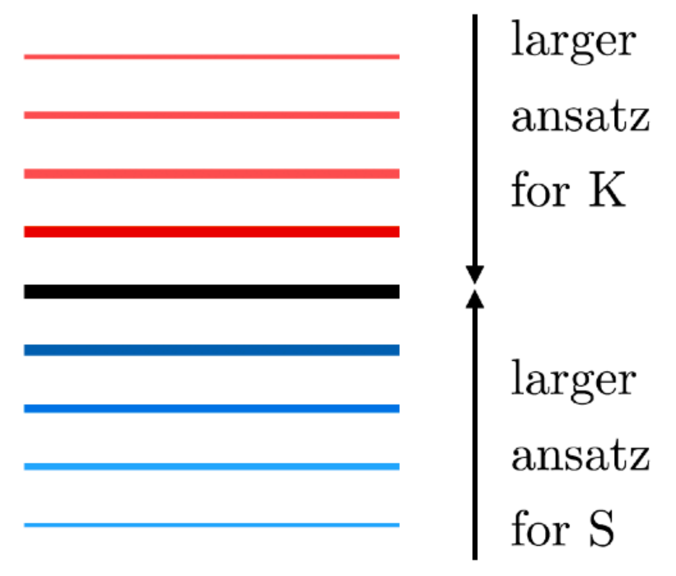
Dual functions K are Lagrange multipliers for S-matrix constraints

Simplest example:

$$\mathscr{F} = S(s_0) = \frac{1}{2\pi i}\oint_{s_0} K(s)S(s) = \frac{2}{\pi}\int_{4m^2}^{\infty} Im[K(s)S(s)] \leq \frac{2}{\pi}\int_{4m^2}^{\infty} |K(s)S(s)| \leq \frac{2}{\pi}\int_{4m^2}^{\infty} |K(s)| = \mathscr{F}_d$$

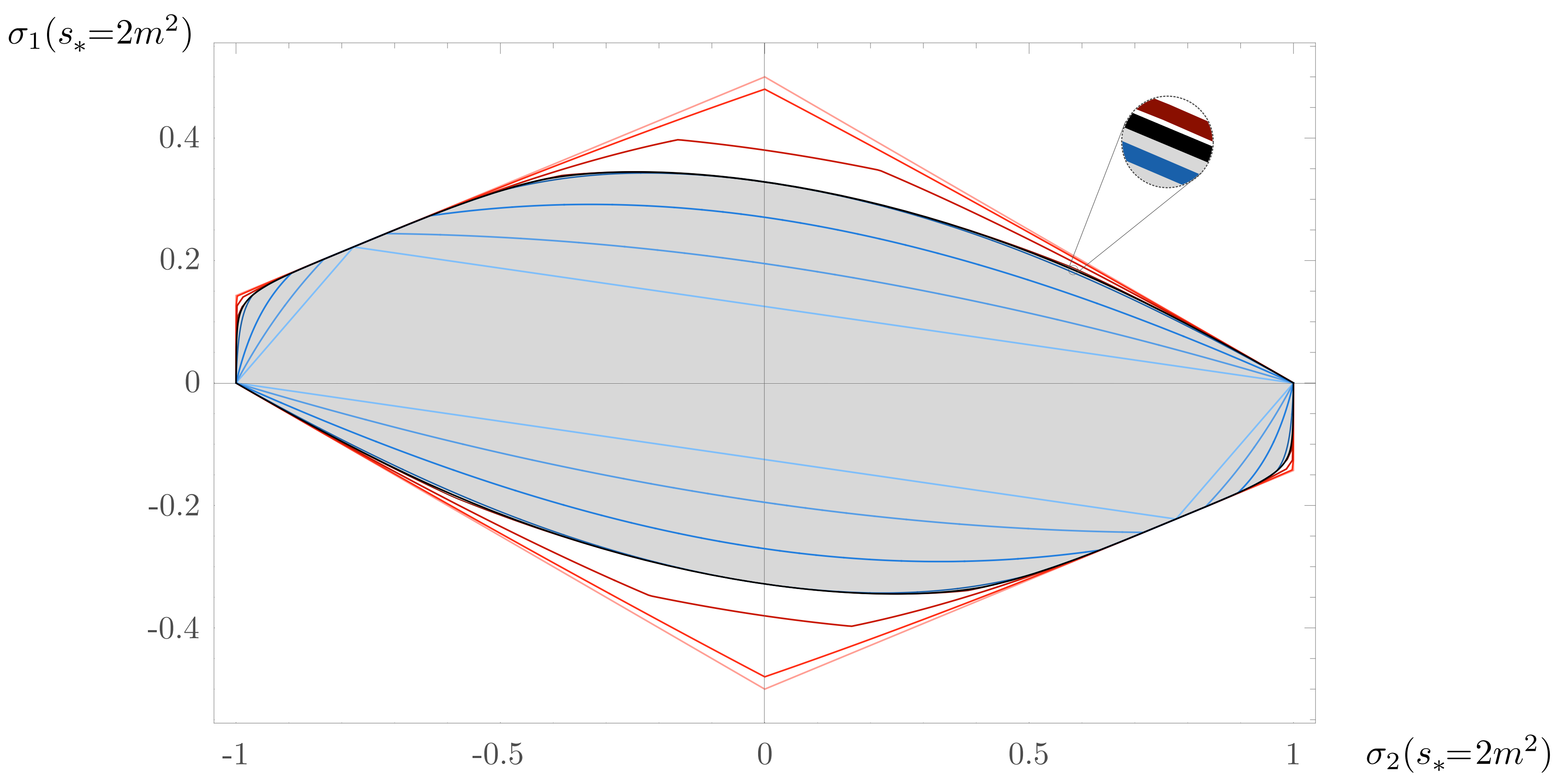
$K(s) \sim \dfrac{1}{s - s_0}$ $\qquad K(4m^2 - s) = -K(s)$ $\qquad |S(s)| \leq 1$

Run Bootstrap for $\mathscr{F}_d[K]$

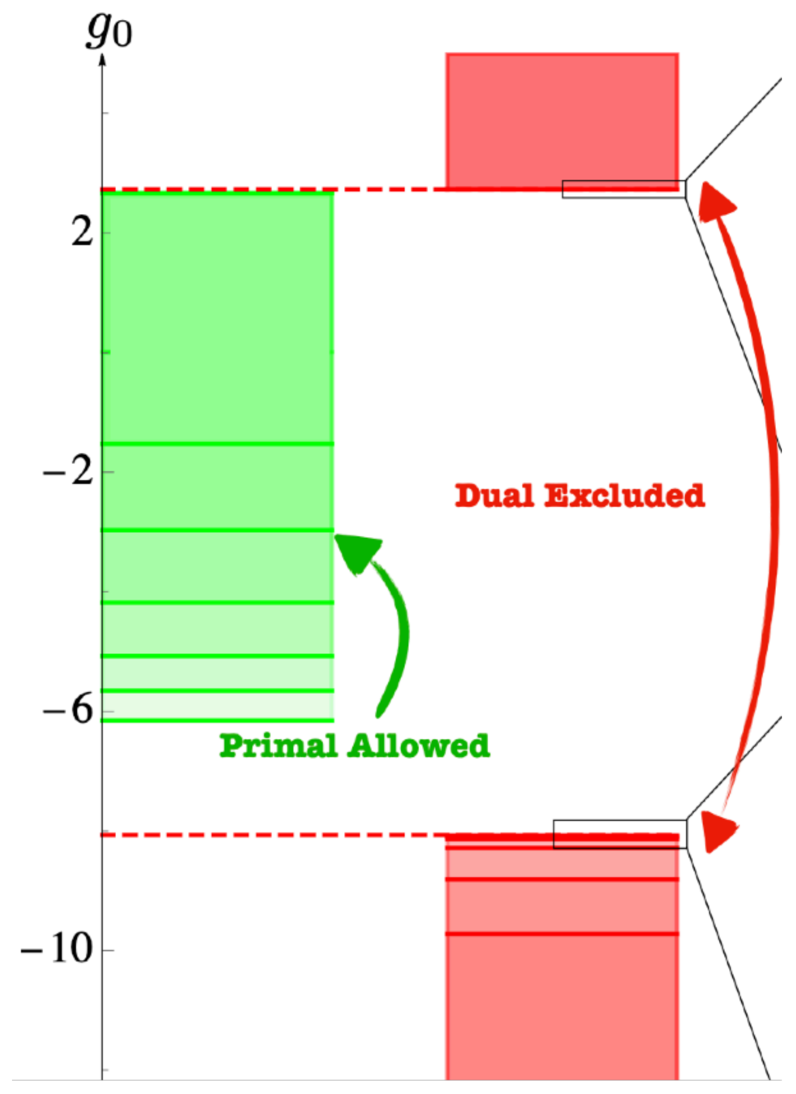


optimality

$$S(s) = \frac{iK^*(s)}{|K(s)|}$$



2D, $O(N)$ sym [LC, He, Kruczenski, Vieira '19]



4D, $T\left(4m^2/3, 4m^2/3\right)$
[Guerrieri, Sever '21]

Applications

Maps in 2D

Two-dimensional scattering is much simpler, but still has a lot to teach us!

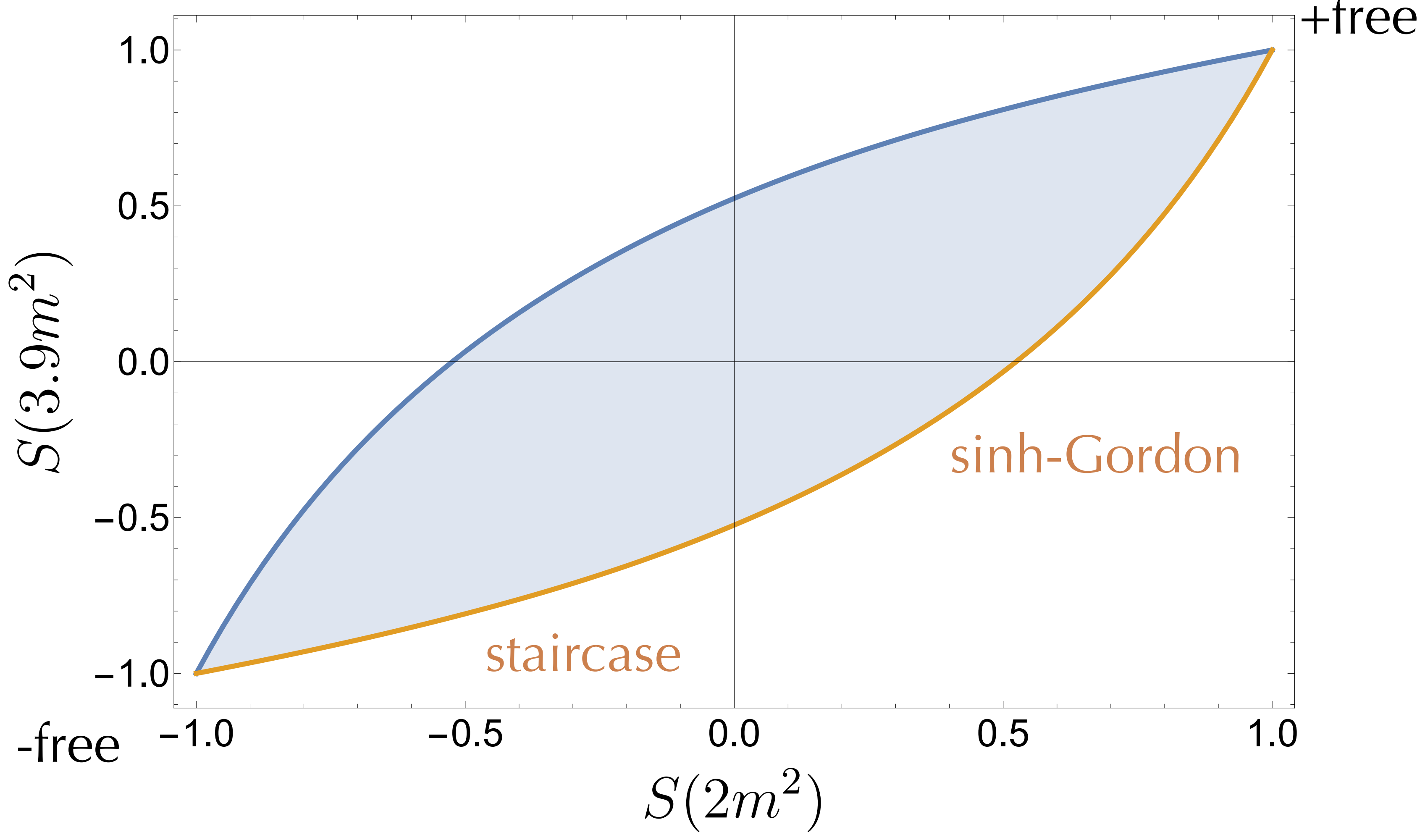
Revisiting warm-up

$S(2m^2)$

-1 $+1$

What if we explore more directions?

See "Leaf" in [Chen, Fitzpatrick, Karateev '21]



Bounds saturated by CDD zeros: resonances

$$S(s) = \pm \frac{\sqrt{s}\sqrt{4m^2 - s} - \sqrt{s_0}\sqrt{4m^2 - s_0}}{\sqrt{s}\sqrt{4m^2 - s} + \sqrt{s_0}\sqrt{4m^2 - s_0}}$$

$$S(s) = S(4m^2 - s) \qquad |S(s)|^2 = 1$$

No particle production → integrable models

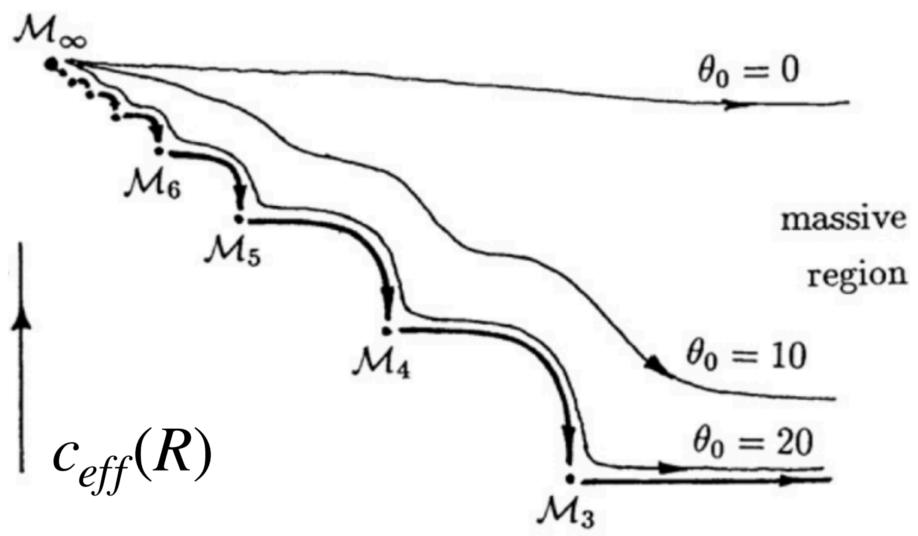
sinh-Gordon

$0 \leq s_0 \leq 4m^2$

$$\mathscr{L}_{int}^{shG} \supset \frac{m_0^2}{\beta^2} \cosh \beta\phi$$

staircase

$s_0 = 2m^2 + i\mathbb{R}$

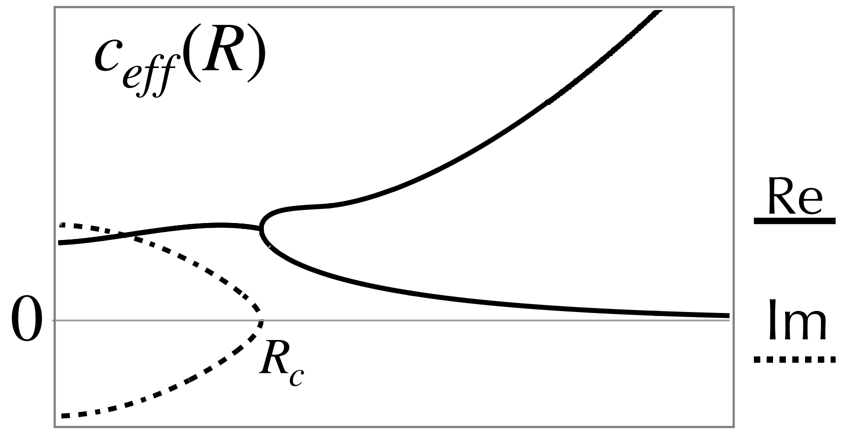


From TBA: [Zamolodchikov '06]

Add more directions → $\prod$ CDD zeros

"Consistent" amplitudes with many resonances
→ unconventional UV behaviour

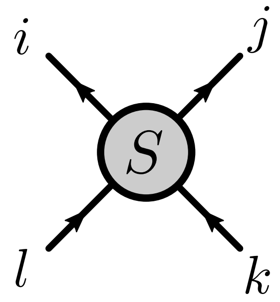
[Camilo, Fleury, Lencses, Negro, Zamolodchikov '21; LC, Negro, Schaposnik Massolo '21]



2D: Global group symmetries

More analytic functions, integrability non-generic

O(N) monolith



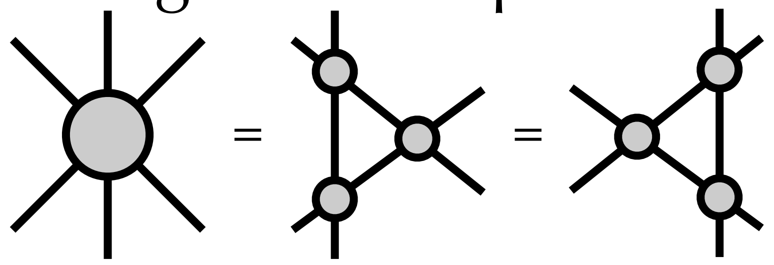
$$S^{ij}_{lk}(s) = \sigma_1(s) + \sigma_2(s) + \sigma_3(s)$$

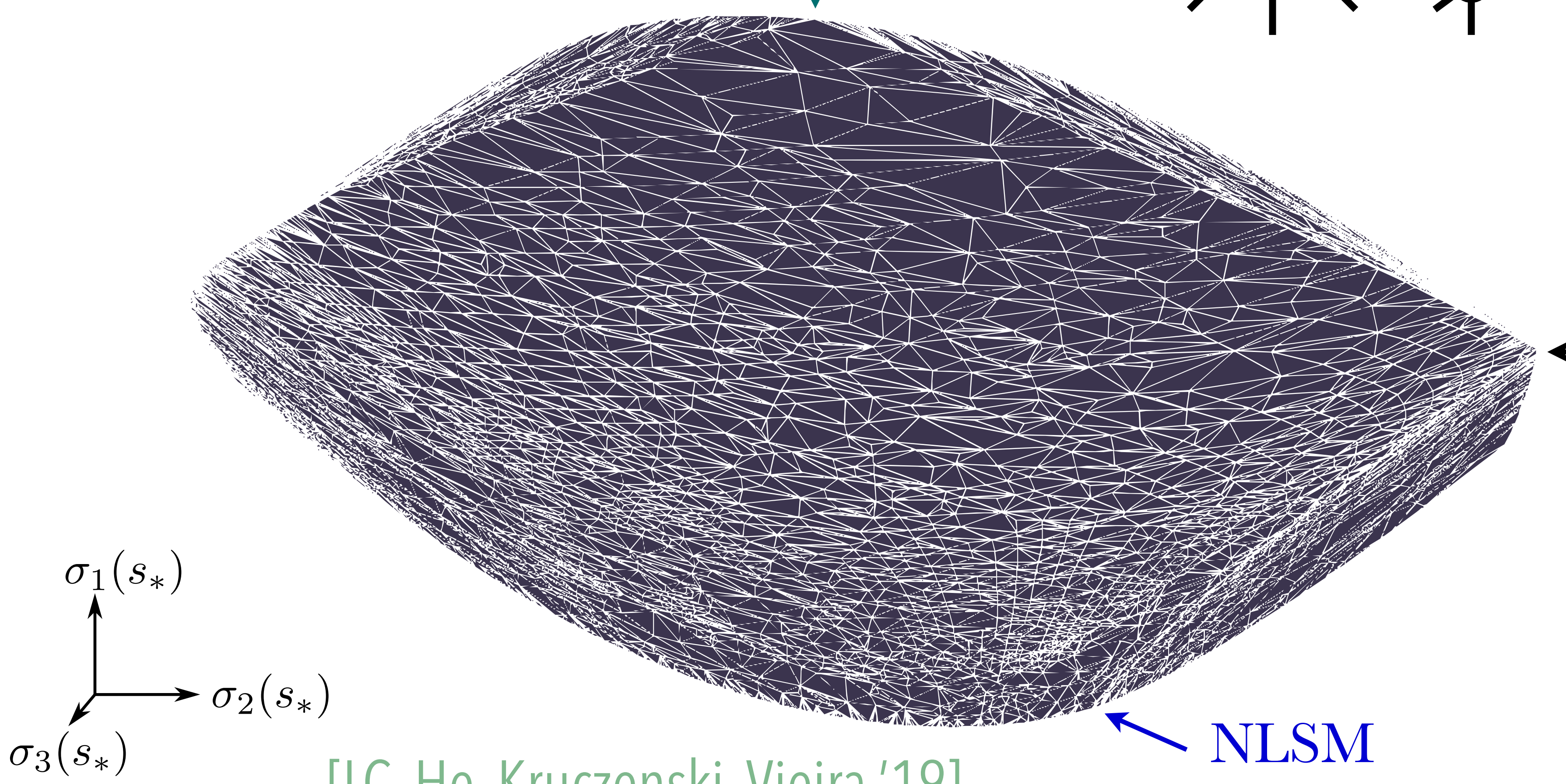
$$= S_\bullet(s)\,(\mathbb{P}_\bullet)^{kl}_{ij} + S_A(s)\,(\mathbb{P}_A)^{kl}_{ij} + S_S(s)\,(\mathbb{P}_S)^{kl}_{ij}$$

Crossing $S_a(4m^2 - s) = \sum_{a'} \mathbf{C}_{a,a'}\, S_{a'}(s)$

Unitarity $|S_a(s^+)| \leq 1$

Yang-Baxter equations

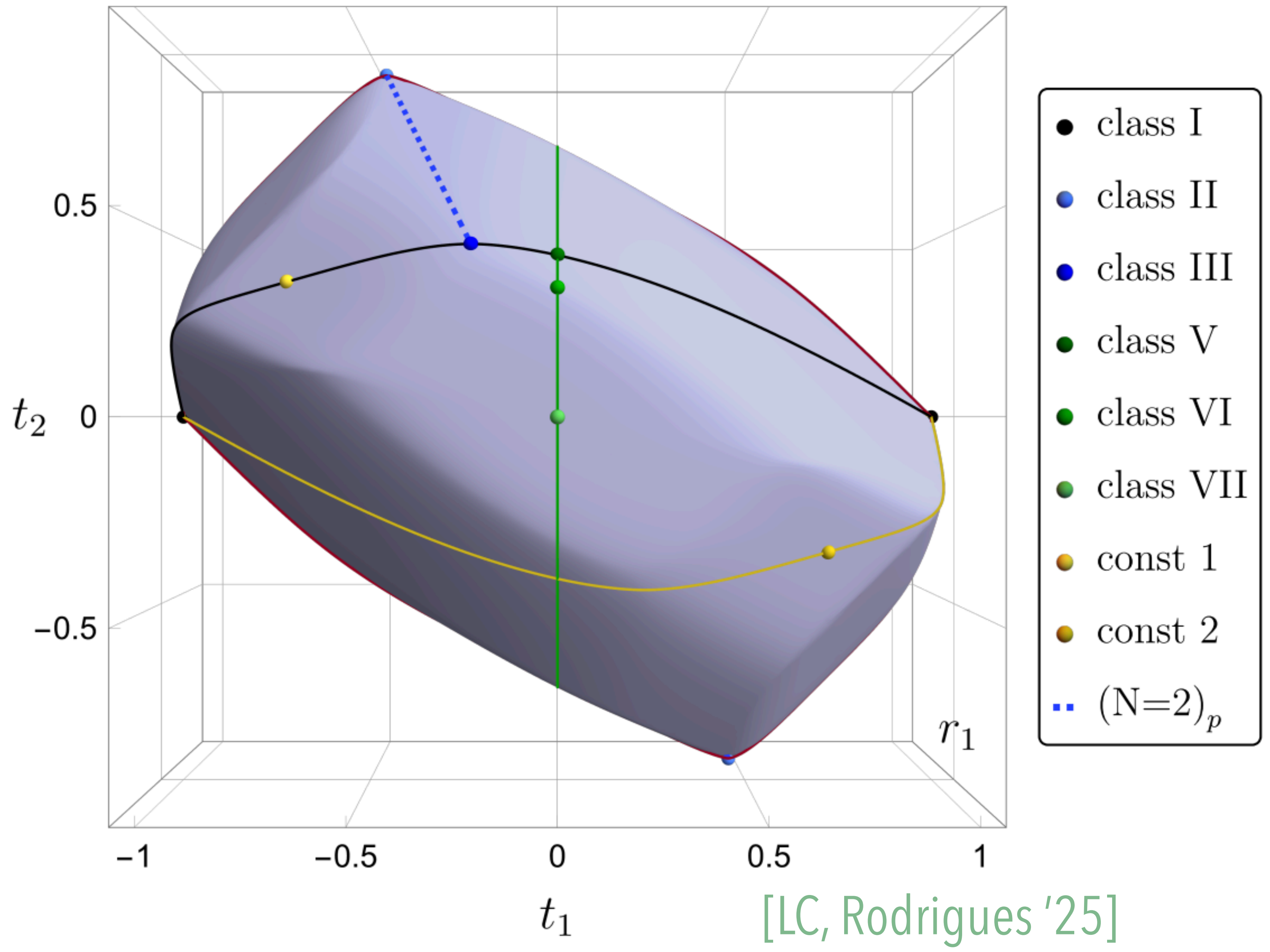




[LC, He, Kruczenski, Vieira '19]

U(N) monolith

6 functions of s: $t_{1,2}$, $r_{1,2}$, $u_{1,2}$



[LC, Rodrigues '25]

- Generic: unitarity saturation (even w/o YB), emergent periodicity in $\theta \sim \ln s$
- Integrable periodic amplitudes show 'walking' behaviour, possible connection to complex CFTs [Gorbenko, Zan '20]

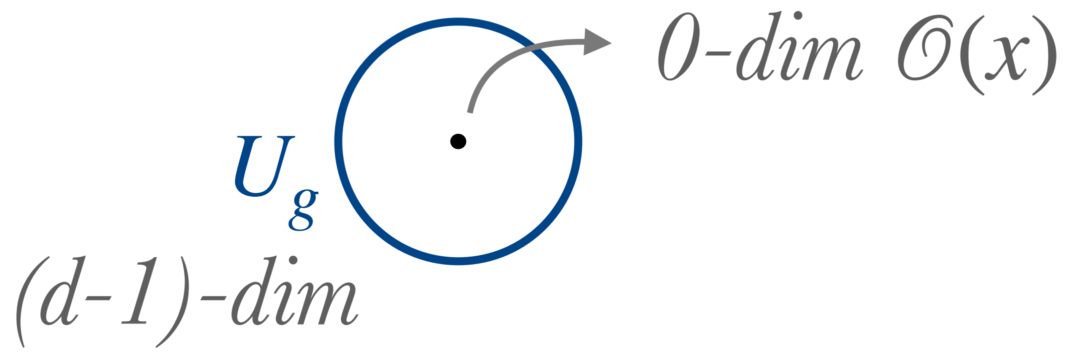
Generalized Symmetries

- (global) Symmetries in QFT ↔ Topological operators [Gaiotto, Kapustin, Seiberg, Willett '14]

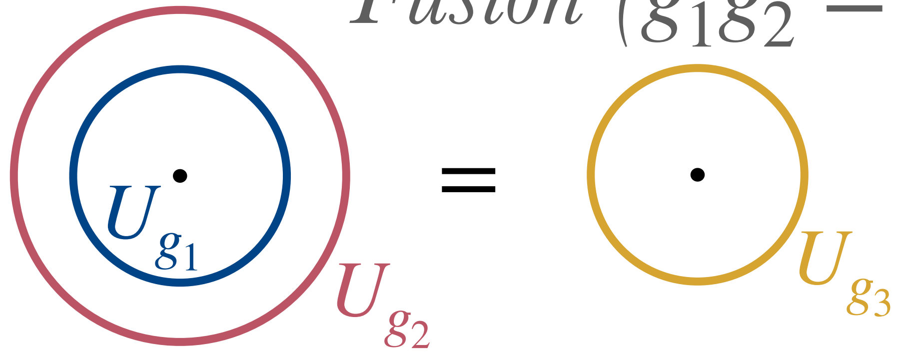
  Move in t → charge conservation

  e.g. from Noether current j_μ: $U = \exp\left(ia\oint d^{d-1}x\, j_0(x)\right)$

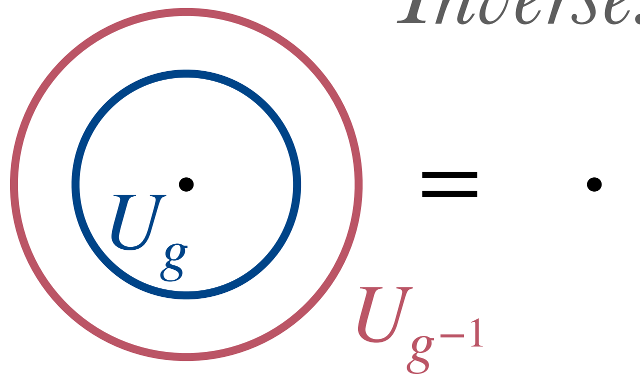
- Usual group symmetry: 0-form, invertible



Fusion $(g_1 g_2 = g_3)$



Inverse: $g^{-1}g = 1$

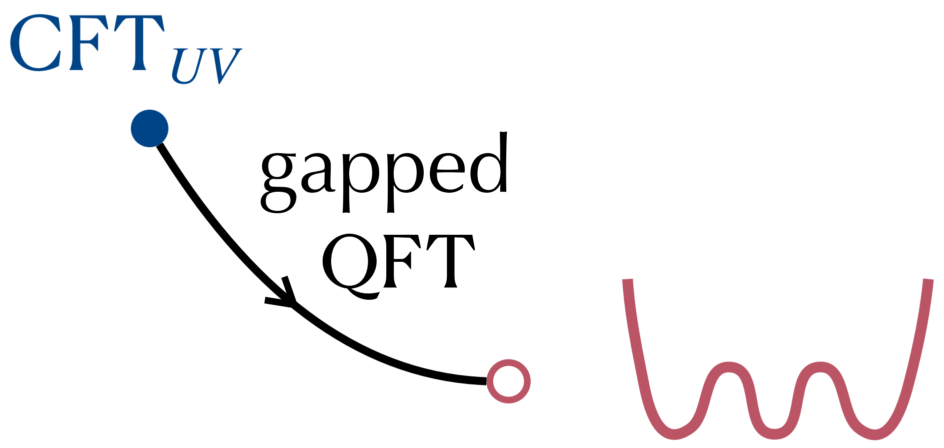


- Generalizations: Higher-form symmetries. *q-form sym, (d-q-1)-dim topological ops.*

  Non-invertible symmetries. g^{-1} *not necessary*

  Here: 0-form symmetries in 2d → topological lines $\mathcal{L}_a$ described by **Fusion categories** (*e.g. Ising*)

- ∃ symmetries and anomalies → strong constraints on IR

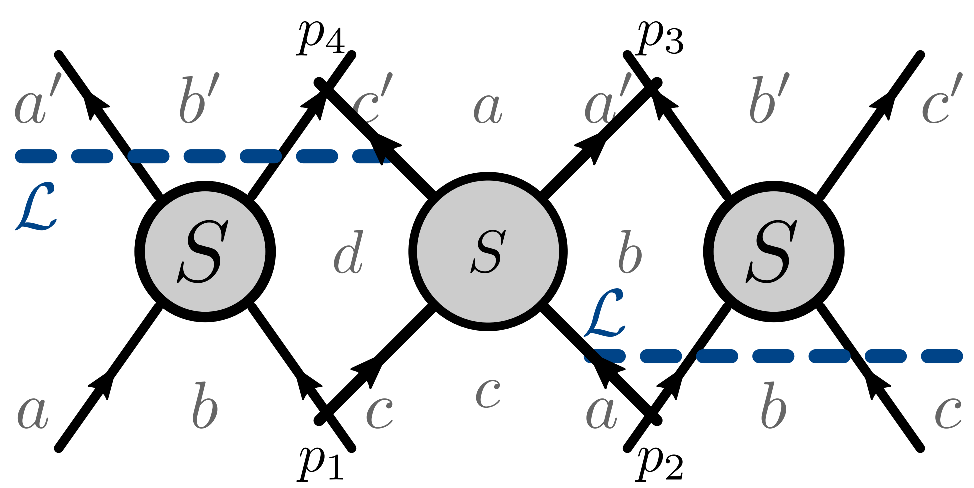


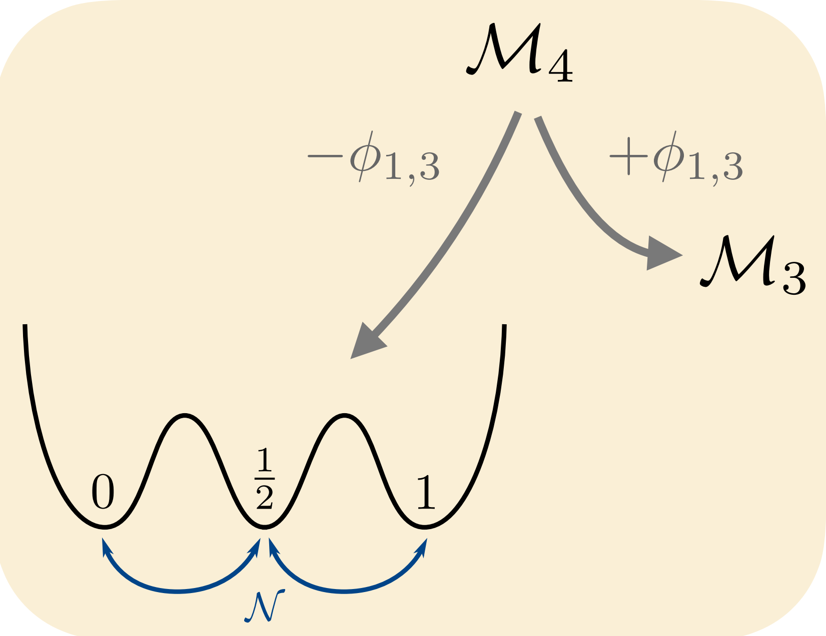
e.g. **degenerate vacua** *in IR*
[Chang, Lin, Shao, Wang, Yin '18]

Non-invertible symmetries and modified crossing

Scattering of kinks interpolating between vacua

Implications of symmetries: **Crossing symmetry of S-matrix gets modified!**





Example: integrable deformation of Tricritical Ising 1+1d

- unitarity + crossing + integrability [Zamolodchikov '89] → ~~non-invertible syms~~

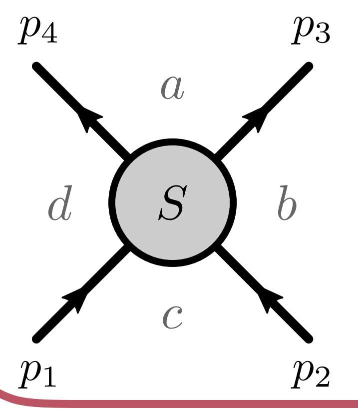
$$\widehat{S}^{ab}_{dc}(\theta) = Z(\theta)\left(\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}\right)^{\frac{i\theta}{2\pi}}\left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\,\sinh\left(\frac{\theta}{n}\right)\delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\delta_{ac}\right]$$

$(s = 4m^2\cosh^2(\theta/2),\ n = 4,\ \mathrm{d}_0 = \mathrm{d}_1 = 1,\ \mathrm{d}_{1/2} = \sqrt{2})$

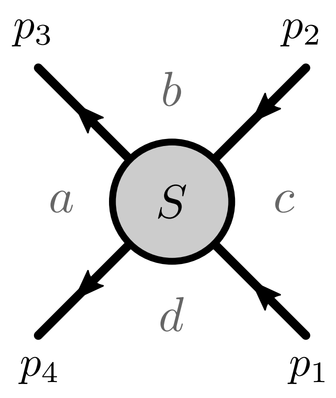
- unitarity + non-invertible syms + integrability

$$S^{ab}_{dc}(\theta) = Z(\theta)\left[\sqrt{\frac{\mathrm{d}_a \mathrm{d}_c}{\mathrm{d}_b \mathrm{d}_d}}\,\sinh\left(\frac{\theta}{n}\right)\delta_{bd} + \sinh\left(\frac{i\pi-\theta}{n}\right)\delta_{ac}\right]$$

→ **Modified crossing** [Copetti, LC, Komatsu '24]

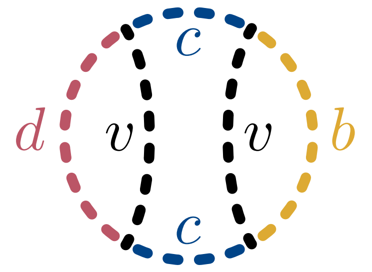


$$S^{ab}_{dc}(s) = \sqrt{\frac{d_a d_c}{d_b d_d}}\; S^{bc}_{ad}(t)$$



symmetry data *(d_a: quantum dimensions)*

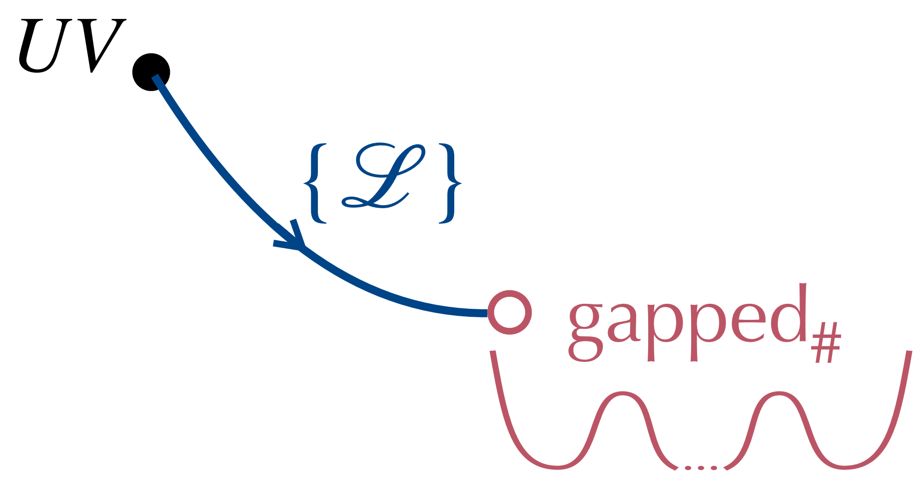
In general: normalization of $|\mathrm{in}\rangle$ and $|\mathrm{out}\rangle$ states, taking into account topological degrees of freedom.

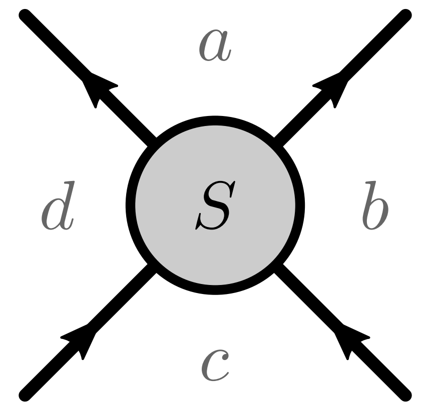
$\langle \mathrm{in}|\mathrm{in}\rangle_s =$  $\times\, \delta^2(\,\cdot\,)$

S-matrix Bootstrap with non-inv. sym.

Set-up:

2D QFT with categorical sym.





$2 \to 2$ Kink scattering

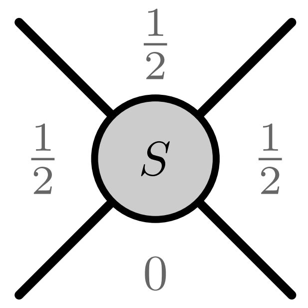
Bootstrap from **A**+**mC**+**U**+**S** assuming:

- minimum spectrum required by symmetry
- SSB all symmetries (#vacua=#$\{\mathscr{L}\}$)

▸ **Symmetries $\{\mathscr{L}\}$** $\quad S^{ab}_{dc}(s) = \sum_{\chi} A_{\chi}(s)(P_{\chi})^{ab}_{dc}$ → Projector into fusion channel χ

e.g. Fibonacci category ($W^2 = 1 + W$)

- 2 vacua $a = 0, 1/2$
- 2 partial amplitudes $A_0(s), A_{1/2}(s)$

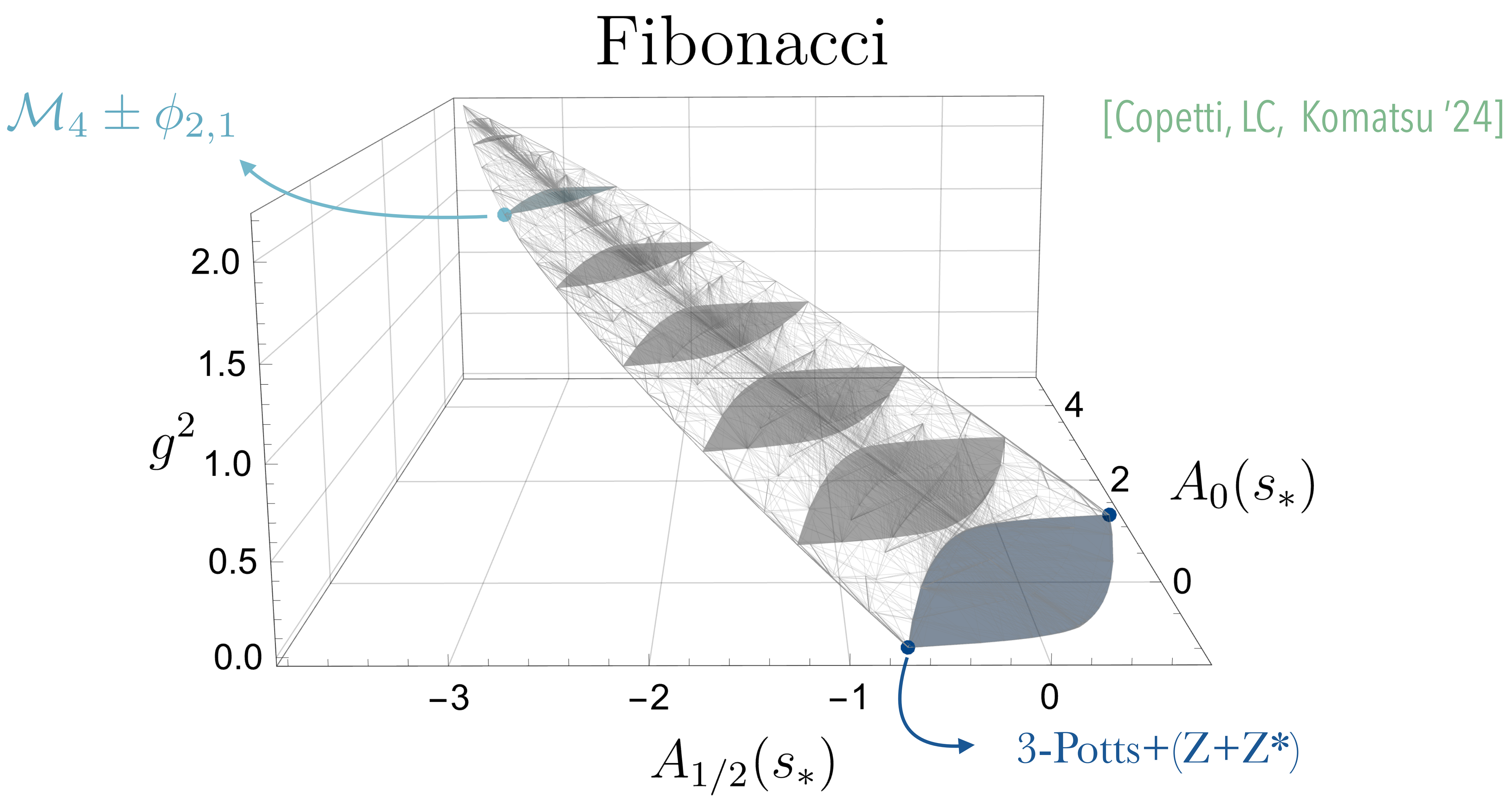


$K_{a,a\pm1/2}$

$K_{1/2,1/2}$

- Cubic coupling $A_{1/2}(s) \sim \frac{g^2}{s-m^2}$

Bootstrap $\{A_0(s_*), A_{1/2}(s_*), g \sim \text{vertex}(\tfrac{1}{2},\tfrac{1}{2},0)\} \implies$



4D: Pions

Pions $\pi^{\pm,0}$ pseudo-Goldstone bosons for chiral symmetry breaking, triplet $SU(2)_V$.

$$\pi_a(p_1) + \pi_b(p_2) \to \pi_c(p_3) + \pi_d(p_4) \qquad T_{ab,cd} = A(s,t,u)\delta_{ab}\delta_{cd} + A(t,s,u)\delta_{ac}\delta_{bd} + A(u,t,s)\delta_{ad}\delta_{bc}$$

Two recent approaches:

Adding UV info through Form Factors

[He, Kruczenski '23-25]

Include currents $\int e^{ipx}\,\mathcal{O}(x)\,|0\rangle$

[Karateev, Kuhn, Penedones '19]

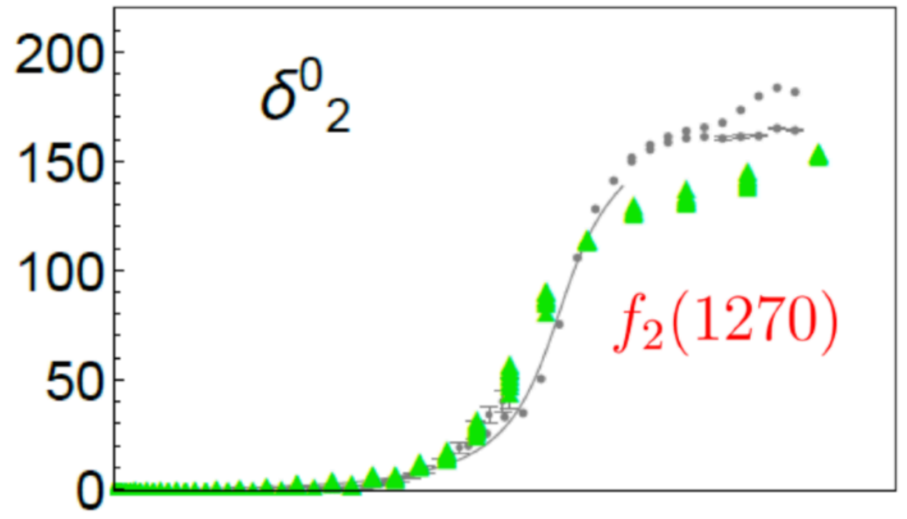
$$\begin{pmatrix} 1 & S^* & F^* \\ S & 1 & F \\ F & F^* & 2\pi\rho \end{pmatrix} \succeq 0$$

Energy

Pert. QCD
FF asymptotics

S-matrix/FF
Bootstrap

Chiral SB



$\Longrightarrow$ Reproduce masses for ρ, ρ', f_2

Fitting to experimental data

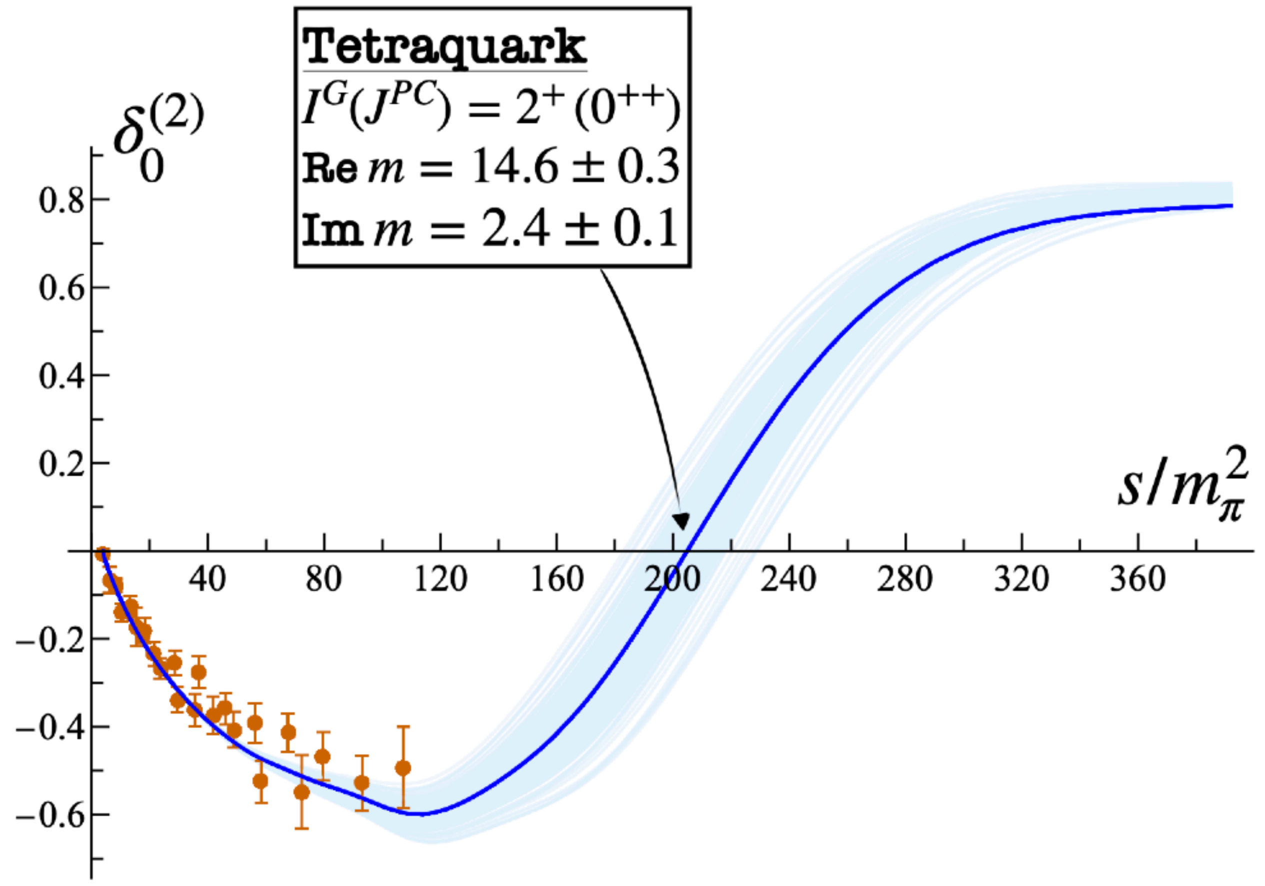
[Guerrieri, Häring, Su '24]

Input e.g. masses for ρ, f_0, f_0', f_2

Energy

S-matrix
Bootstrap

Fit to data



$\Longrightarrow$ Predict new resonances

Final Remarks

Final Remarks

S-matrix Bootstrap

A very useful tool to carve out the space of consistent QFTs/learn more about your favourite theory

Many applications I did not talk about: e.g. Effective Field Theories and Gravity (see rest of the conference)

Some future directions I am interested in :

- Beyond lightest $2 \to 2$ scattering

  Multiparticle Flux-Tube [Guerrieri, Homrich, Vieira '24]
  Non-perturbative anomalous thresholds [Correia '22]

- Axioms not fully proven, might need modifications

- 4D gauge and gravity: IR finite observables for long-range interactions?

Thank you!