Politopos de Fano y reflexivos en las integrales de Feynman

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Based on 2512.10518 [hep-th] with Pavel Novichkov and Pierre Vanhove

New Trends in QFT, Amplitudes and Gravity
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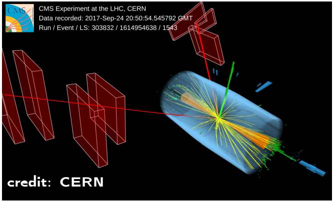
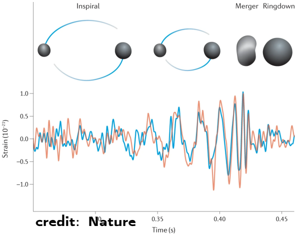
Contents

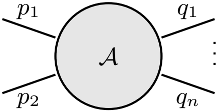
Basic problem



Scattering at LHC



Waveform

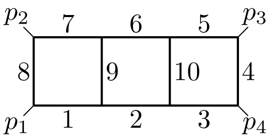


Generalized unitarity

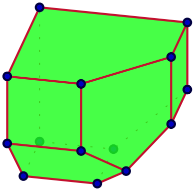
$$\mathcal{A}_n = \sum_i c_i I_i^{\text{basis}} + \text{rational} \downarrow$$

Bottleneck: Evaluation of Feynman integrals

Feynman integral + polytopes



Feynman integrals



Newton polytopes

1. Sector decomposition SecDec 3.0, '15
2. Feynman integrals in Lee-Pomeransky representation Mint, '13 and Euler characteristic Bitoun et'al, '18
3. Evaluation via A-hypergeometric functions LDLC, '19
4. Landau Analysis Klausen '21, Fevola et'al 23'
5. Method of regions Smirnov-Pak, '10
6. Finite integrals
7. Fano and Reflexivity

Today:

Feynman integrals and Fano and reflexive polytopes

Feynman integrals in loop representation

- Feynman integral in dim-reg $(D = 4 - 2\epsilon)$

$$\hat{I}[\mathcal{N}(\boldsymbol{\ell}, \boldsymbol{k})] = \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{i\pi^{D/2}} \, \frac{\mathcal{N}(\boldsymbol{\ell}, \boldsymbol{k})}{\mathcal{D}_1 \cdots \mathcal{D}_E}$$

- Denominators

$$\mathcal{D}_e = (M_e)^{jr} \ell_j \cdot \ell_r + 2(Q_e)^{jr} \ell_j \cdot k_r + J_e + i\varepsilon$$

- Numerators $\mathcal{N}(\boldsymbol{\ell}, \boldsymbol{k})$:
  1. Polynomials in $W \equiv \{\, \ell_i \cdot \ell_j,\, \ell_i \cdot k_j \,\}$
  2. Highest rank r
  3. We impose momentum conservation on external momenta $\boldsymbol{k}$

Example: $L = 2$

$$\mathcal{N}(\boldsymbol{\ell}, \boldsymbol{k}) = c_1 k_1 \cdot k_2 + c_2 \ell_2^2 + c_3 (\ell_1 \cdot k_1)^2 + \cdots + c_n (\ell_1 \cdot \ell_2)^2$$

Highest rank is $r = 4$

Symanzik polynomials

- Construct matrices (Feynman parameters z_e)

$$\widetilde{M}^{jr} = \sum_{e=1}^{E} z_e M_e^{jr}, \quad \widetilde{Q}^{j\mu} = \sum_{e=1}^{E} z_e Q_e^{jr} k_r^{\mu} \quad \widetilde{J} = \sum_{e=1}^{E} z_e J_e$$

- Symanzik polynomials

$$\mathcal{U} = \det(\widetilde{M}), \quad \mathcal{F} = \det(\widetilde{M}) \left(\widetilde{J} - \left(\widetilde{M}^{-1}\right)^{ij} \widetilde{Q}^i \cdot \widetilde{Q}^j \right) / \mu^2$$

Symanzik polynomials

- In Euclidean kinematics: invariants $-(p_i + p_j)^2 > 0$
- Kinematic dependence is in $\mathcal{F}$
- $\mathcal{U}$ homogeneous polynomial of degree L
- $\mathcal{F}$ homogeneous polynomial of degree $L + 1$
- $\mathcal{U}$, $\mathcal{F}$ positive functions of their parameters
- $\mathcal{U}$ and $\mathcal{F}$ can only vanish on the boundaries of the integration region

Feynman integrals in parametric representation

Parametric representation

$$\hat{I}[\mathcal{N}(\boldsymbol{\ell},\boldsymbol{k})] = \Gamma\left(E - \left\lfloor \frac{r}{2} \right\rfloor - \frac{LD}{2}\right) \int \mathrm{d}^E z \, \delta\left(1 - \sum_{i \in A} z_e\right) \mathcal{U}^{E-D/2(L+1)-r} \mathcal{F}^{DL/2-E} \widetilde{\mathcal{N}}(\boldsymbol{z})$$

- $\lfloor r/2 \rfloor$ denotes the nearest integer less or equal to r.
- $\deg(\widetilde{\mathcal{N}}(\boldsymbol{z})) = rL$
- The numerator is a polynomial in E variables

$$\widetilde{\mathcal{N}}(\boldsymbol{z}) = \sum_i c_i \boldsymbol{z}^{\boldsymbol{n}_i}$$

exponent vectors $\in \mathbb{N}^E$

Example

$$\widetilde{\mathcal{N}}(\boldsymbol{z}) = -\left(k_1^2 z_1 z_3 + k_3^2 z_2 z_3\right)$$

$$\boldsymbol{n}_1 = (1,0,1), \boldsymbol{n}_2 = (0,1,1)$$

Cheng-Wu

- Cheng-Wu theorem allows us to choose A to be any nonempty subset of $\{1,\ldots,E\}$
- We choose the last edge $A=\{E\}$

$$\hat{I}[\mathcal{N}(\boldsymbol{\ell},\boldsymbol{k})]=\Gamma\left(E-\left\lfloor\frac{r}{2}\right\rfloor-\frac{LD}{2}\right)\int \mathrm{d}^{E-1}z\,\mathcal{U}^{E-D/2(L+1)-r}\mathcal{F}^{DL/2-E}\widetilde{\mathcal{N}}(\boldsymbol{z})$$

- $\widetilde{\mathcal{N}}(\boldsymbol{z})$ is not homogeneous anymore

$$\widetilde{\mathcal{N}}(\boldsymbol{z})=\sum_i c_i\boldsymbol{z}^{\boldsymbol{n}_i}$$

exponent vectors $\in \mathbb{N}^{E-1}$

- We will consider each monomial at the time

Newton Polytopes

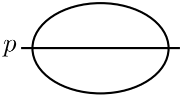
- For a polynomial in $E-1$ variables in some monomials basis B

$$f(\boldsymbol{z}) = \sum_{\boldsymbol{m}\in B} c_{\boldsymbol{m}} \boldsymbol{z}^{\boldsymbol{m}}$$

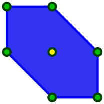
- Newton polytope is the convex hull of its exponent vectors

$$\text{Newton}(f) = \mathrm{conv}(\mathrm{supp}(f)) = \text{ConvexHull}(\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{|B|}) \in \mathbb{R}^{E-1}$$

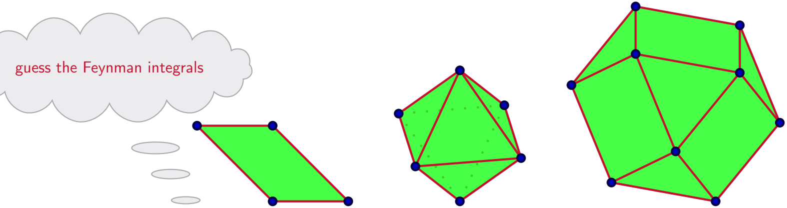
- Sunset



$$\mathrm{Newt}(\mathcal{F}) = \mathrm{Newt}(m^2(z_1z_2 + z_2 + z_1)(z_1 + z_2 + 1) - p^2 z_1 z_2)$$



Representations of polytopes



- V-representation: finite set of extreme points

$$P = \text{conv}\,\{\,p_1, p_2, \ldots\,\}$$

- H-representation: M inequalities

$$P = \{\,\boldsymbol{m} \in \mathbb{Z}^E \mid A\boldsymbol{m} - b \geq 0\,\}$$

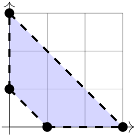
Properties of Newton polytopes

proper or improper

1. $\text{face}_w(P) \equiv \{\boldsymbol{m} \in P | w \cdot \boldsymbol{m} \geq w \cdot v \quad \text{for all } v \in P\}$
2. *relative interior* of a polytope P: the polytope with its proper faces removed
3. Minkowski sum: $P + Q \equiv \{\, p + q \mid p \in P, q \in Q \,\} \subset \mathbb{R}^E$
4. scalar multiplication: $\lambda P \equiv \{\, \lambda p \mid p \in P \,\}\ , \quad \lambda \in \mathbb{R}$
5. f, g polynomials, $n > 0$

$$\begin{aligned} \text{Newt}(fg) &= \text{Newt}(f) + \text{Newt}(g) \\ \text{Newt}(f^n) &= n\,\text{Newt}(f) \end{aligned}$$

Example:



V : Conv$\{(0,1),(1,0),(0,2),(2,0)\}$

$$H : \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ -2 \end{pmatrix} \geq 0$$

Convergence of tensor integrals

- Feynman integrals are Euler-Mellin integrals studied by Berkesh-Forsgård-Passare, '13
- One monomial at the time

$$\hat{I}[\mathcal{N}(\boldsymbol{\ell},\boldsymbol{k})] \sim \sum c_i \int \mathrm{d}\eta_{E-1} \, \frac{z^{\boldsymbol{m}_i+1}}{\mathcal{U}^{r-E+D/2(L+1)}\mathcal{F}^{E-DL/2}}, \quad \mathrm{d}\eta_{E-1} = \frac{\mathrm{d}z_1}{z_1}\cdots\frac{\mathrm{d}z_{E-1}}{z_{E-1}}$$

Theorem (Berkesh-Forsgård-Passare)

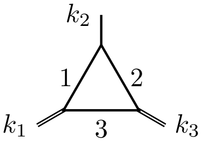
- Let $n_{\mathcal{U}} \equiv r - E + D/2(L+1), \quad n_{\mathcal{F}} \equiv E - DL/2$
- Assume that Symanzik polynomials are non-vanishing on region of integration, full dimensional $n_{\mathcal{U}} > 0$, $n_{\mathcal{F}} > 0$,
- Feynman polytope (full-dimensional)

$$P_F = n_{\mathcal{U}}\,\mathrm{Newt}(\mathcal{U}) + n_{\mathcal{F}}\,\mathrm{Newt}(\mathcal{F})$$

- BFP: The integral converges and defines an analytic function in the Mandelstam invariants and in the domain for the exponents

$$\{\, \boldsymbol{m} \in \mathbb{C}^E \mid \boldsymbol{m} + \mathbf{1} \in \mathrm{relint}(P_F) \,\}$$

Toy example: rank-1 finite numerator in four dimensions



- Kinematics

$$k_1 \neq k_3, \quad k_1^2 \neq 0, k_3^2 \neq 0, \quad k_2^2 = 0$$

- Parametric representation

$$\hat{I}_{\triangle,r} = \sum_{\boldsymbol{m}\in B} c_{\boldsymbol{m}}(\boldsymbol{s}) \int \frac{\mathrm{d}^3 z \, z^{\boldsymbol{m}+1}}{z_1 z_2 z_3} \, \frac{\delta(1 - \sum_i z_i)}{\mathcal{U}^{-3+D+r} \mathcal{F}^{-D/2+3}}$$

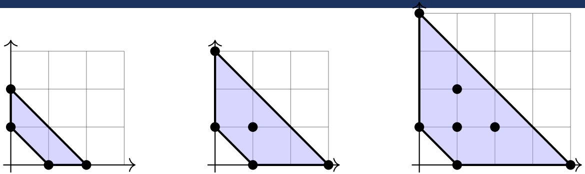
- Symanzik polynomials

$$\mathcal{U} = z_1 + z_2 + z_3 \qquad \mathcal{F} = -(k_1^2 z_1 z_3 + k_3^2 z_2 z_3)$$

- Feynman polytope

$$P_\triangle = (r + D - 3)\,\mathrm{Newt}(\mathcal{U}) + (3 - D/2)\,\mathrm{Newt}(\mathcal{F})$$

Toy example: rank-1 finite numerator in four dimensions



- Set $r = 1$

$$\boldsymbol{m} + 1 = (1,1) \Rightarrow \boldsymbol{m} = \boldsymbol{0} \Rightarrow (z_1^0 z_2^0) \quad \text{is a "finite monomial"}$$

- General numerator of rank 1

$$\mathcal{N}(\ell, \boldsymbol{k}) = c_1 \ell \cdot k_1 + c_2 \ell \cdot k_2$$

free coefficients c_i

- In parametric representation

$$\mathcal{N}(z, \boldsymbol{k}) = \frac{1}{2} c_1 \left(k_3^2 - k_1^2\right) z_2 - c_1 k_1^2 + \frac{1}{2} c_2 \left(k_1^2 - k_3^2\right)$$

- Coefficients of non-finite monomials must vanish

$$c_1 = 0 \Rightarrow \mathcal{N} = \ell \cdot k_2 \text{ is a rank-1 finite numerator}$$

Full story: **Finite integrals from Feynman polytopes**, DLC-Kosower-Novichkov, *Phys. Rev. D* ***111*** *(2025) no.10, 105013*

Fano and reflexive polytopes

Fano polytope Nill '05, Kasprzyk '12

A lattice polytope is (canonical) Fano if the only lattice point that lies strictly in its interior is the origin.

- Let N be the dual lattice to M with respect to the scalar product $\langle \bullet | \bullet \rangle : M \times N \to \mathbb{Z}$
- Using this scalar product we can define the dual (or polar) polytope

$$\nabla := \{\boldsymbol{b} = (b_1, \ldots, b_n) \in N \mid \langle a, b \rangle \geq -1, \, \forall \boldsymbol{a} \in \Delta\} \subset N \cong \mathbb{Z}^n$$

Reflexive polytope

A lattice polytope Δ is reflexive if its polar polytope ∇ is a lattice polytope and has a single interior point. The polytopes (Δ, ∇) are said to be mirror pairs.

Reflexive polytopes

- Reflexive polytopes appear in String Theory to construct Calabi-Yau varieties
- Classified up to 4-dimensions Maximilian Kreuzer and Harald Skarke

  Complete classification of reflexive polyhedra in four-dimensions, Kreuzer-Skarke, *Adv. Theor. Math. Phys. 4 (2000) 1209*

- There are 473,800,776 reflexive polyhedra in four dimensions

Observation

Feynman polytope with a single interior point $\boldsymbol{p}$ is finite if it satisfies BFP theorem

$$I_F \sim \int \mathrm{d}\eta_{E-1} \frac{z^{\boldsymbol{p}}}{\mathcal{U}^{n_\mathcal{U}} \mathcal{F}^{n_\mathcal{F}}}, \qquad \mathrm{d}\eta_k := \frac{\mathrm{d}z_1}{z_1} \cdots \frac{\mathrm{d}z_k}{z_k}$$

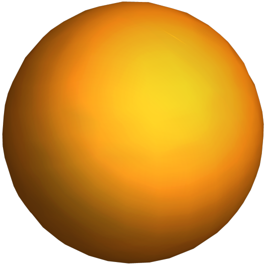
Question:

Is there a way to determine how many polytopes are reflexive for a given Feynman graphs and/or for all Feynman graphs?

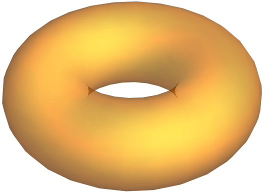
What on earth are Calabi-Yau varieties?

- In cartoons!

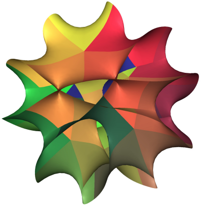


1 loop



2-loop and beyond



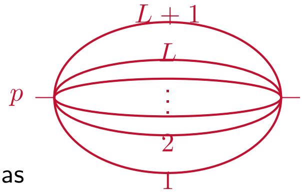
2-loop and beyond

Banana integrals: the Verrill class of polytopes

- Banana integrals in two dimension are the basic examples where reflexive polytopes arise

$$\mathcal{U}_L = z_1 z_2 \cdots z_{L+1} \left(\frac{1}{z_1} + \cdots + \frac{1}{z_{L+1}} \right)$$

$$\mathcal{F}_L = \mathcal{U}_L \left(m_1^2 z_1 + \cdots + m_{L+1}^2 z_{L+1} \right) - s z_1 \cdots z_{L+1}$$

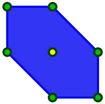


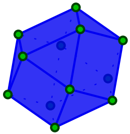
- For banana integrals $N = L+1$ and in $d = 2$ one has

$$P_{\mathsf{F}} = \mathrm{Newt}(\mathcal{F})$$

- The classical period

$$I_L \sim \int \mathrm{d}\eta_L \frac{1}{1 - t\phi_L(x)}$$





Reflexive polytopes for the sunset at 2 and 3-loop

How many are there?

- Inspired by Kreuzer-Skarke[also Fanosearch project]: fix number of dimensions $\sim$ edges

$N = \dim + 1$	graph topologies	Fano classes	reflexive classes	non-reflexive Fano classes
2	1	1	1	0
3	2	2	2	0
4	3	3	3	0
5	6	4	4	1
6	13	8	7	4
7	28	11	6	6
8	70	23	11	16
9	193	36	14	24
10	565	104	26	88

N-gon class: Calabi-Yau ?

- For $L = 1$ the polytope is the scaled standard simplex

$$N\Delta_{HS}^{N,1}$$

- The polar polytope the polar polytope has the simple expression

$$P_F^\circ = \text{Conv}\Big\{e_1, e_2, \ldots, e_{N-1}, -\sum_{i=1}^{N-1} e_i\Big\}$$

- Nonvanishing Hodge numbers for N-gon. In all cases $h_{11} = 1$.

N	4	5	6	7	8	9	10
h_{1N-3}	19	101	426	1667	6371	24229	92278

- Encodes projective space $\mathbb{P}^{N-1}$
- Quintic $N = 5$

$$P_F = \text{Conv}\Big\{(-1,-1,-1,-1), (4,-1,-1,-1), (-1,4,-1,-1), (-1,-1,4,-1), (-1,-1,-1,4)\Big\}$$

Singular CYs

- 1-loop graphs are expected to evaluate in terms of polylogs so the surface is degenerate
- Example (y_{ij} are dual momentum variables)

$$\phi_{\triangle} = z_1 y_{21}^2 + z_2 y_{21}^2 + \frac{z_1 y_{31}^2}{z_2} + \frac{y_{31}^2}{z_2} + \frac{z_2 y_{32}^2}{z_1} + \frac{y_{32}^2}{z_1}$$

$$\phi_{\text{sunset}} = m_1^2 z_1 + m_2^2 z_2 + \frac{m_2^2 z_2}{z_1} + \frac{m_1^2 z_1}{z_2} + \frac{m_3^2}{z_1} + \frac{m_3^2}{z_2}$$

- Naively they are both elliptic, however

$$1/t_{\triangle} = y_{12}^2 + y_{13}^2 + y_{23}^2, \quad 1/t_{\text{sunset}} = s - (m_1^2 + m_2^2 + m_3^2)$$

- This can also be compared against the vacuum case of the sunset, see section 3 of Ref.[Bloch et'al, '2013].

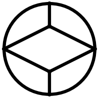
One take home message

Polytopes can teach us about the hidden geometry of Feynman integrals

Evaluation of integrals

- We observe that integrals with a reflexive polytope are simple to evaluate

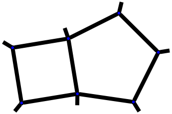
The wheel with three spokes ($6\zeta(3)$)



The diamond circle graph ($36\zeta(3)^2$)

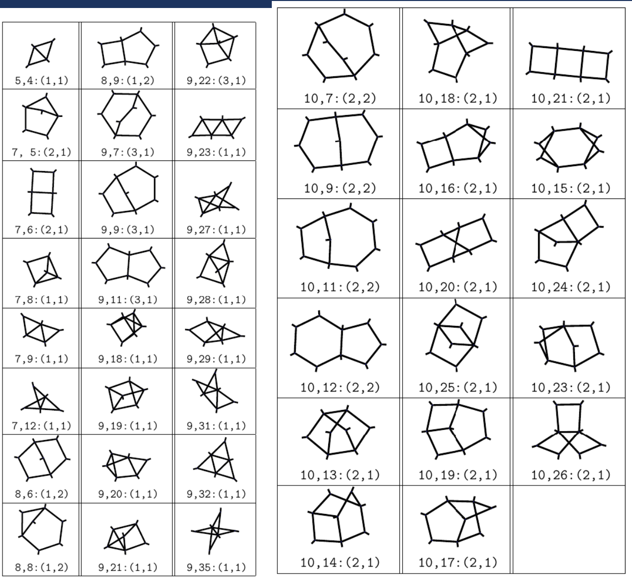
Two graphs that evaluate to zeta values

- Integrals which have associated a Fano polytope are not, for example



- So we gave up quickly

The juice is in Fano cases (9 and 10 edges)



Database: https://github.com/pierrevanhove/ReflexiveFanoPolytopes

Summary and Outlook

Summary

- Convergence of Feynman integrals is controlled by math's theorem
  Berkesch-Forsgård-Passare, 13'
- We can construct systematically locally finite numerators in parametric and loop space
- A new class of reflexive polytopes arise in 1-loop N-gons (unexpected Calabi-Yau geometries)
- Reflexive and Fano polytopes arising from Finite Feynman integrals are sparse

Outlook

- Finite integrals with certain Fano geometries as bases
- Machine Learning finiteness and reflexivity

Thank you!