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One-loop Amplitudes in QED: A Worldline Formalism Approach

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New Trends in Quantum Field Theory,

Amplitudes and Gravity

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One-loop Amplitudes in QED: A Worldline Formalism Approach



Outline:

- QED History & Path Integrals
- The Master Formula
- Integration problem
- Low energy limit
- Conclusions

A Snapshot of QED History

QED History & Path Integrals



The Birth of Positron Theory

○The quantum theory of the electron.

Dirac, P. A. M. (1928).

○Experimental discovery of the positron.

Anderson, C. D. (1933).

○Solvay Conference (1933): Dirac identifies negative energy solutions with positrons.

○Heisenberg: Formalizes quantum fluctuations in the Dirac Sea picture.

Heisenberg, W. (1934).

QED History & Path Integrals



The Birth of QED

- Computation of the light-by-light scattering cross-section in the low energy limit Euler, H., & Kockel, B. (1935)

The Euler-Kockel paper established the conceptual framework of the quantum vacuum as a polarizable medium

- Heisenberg and Euler obtained the full nonlinear correction to the Maxwell Lagrangian

Heisenberg, W., & Euler, H. (1936)

$$\mathcal{L} = \frac{e^2}{hc} \int_0^\infty \frac{d\eta}{\eta^3} e^{-\eta} \left\{ i\eta^2 (\vec{E} \cdot \vec{B}) \frac{\left[\cos \left(\frac{\eta}{\mathcal{E}_c} \sqrt{\vec{E}^2 - \vec{B}^2 + 2i(\vec{E} \cdot \vec{B})} \right) + \text{c.c.} \right]}{\left[\cos \left(\frac{\eta}{\mathcal{E}_c} \sqrt{\vec{E}^2 - \vec{B}^2 + 2i(\vec{E} \cdot \vec{B})} \right) - \text{c.c.} \right]} + \mathcal{E}_c^2 + \frac{\eta^2}{3} (\vec{B}^2 - \vec{E}^2) \right\}$$

- Soon after, Weisskopf derived the equivalent effective action for scalar QED

Weisskopf, V. (1936)

QED History & Path Integrals



The Birth of QED

- Heisenberg and Euler expressed their Lagrangian in terms of a critical field strength

$$\mathcal{E}_c = \frac{m^2 c^3}{e \hbar} \approx 10^{16} \text{ V/cm}$$

- Effective Lagrangian in Spinor and Scalar QED

$$\mathcal{L}_{spin}^{EH} = -\frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left(\frac{e^2 ab}{\tanh(eaT) \tan(ebT)} - \frac{e^2}{3} (a^2 - b^2) - \frac{1}{T^2} \right)$$

$$\mathcal{L}_{scalar}^{EH} = \frac{1}{16\pi^2} \int_0^\infty \frac{dT}{T} e^{-m^2 T} \left(\frac{e^2 abT}{\sinh(eaT) \sin(ebT)} + \frac{e^2}{6} (a^2 - b^2) - \frac{1}{T^2} \right)$$

- Where

$$ab = \vec{E} \cdot \vec{B},$$

$$a^2 - b^2 = \vec{E}^2 - \vec{B}^2,$$

QED History & Path Integrals



Although QED vacuum effects are extraordinarily small, they have been confirmed with unprecedented precision:

- **Electron Anomalous Magnetic Moment ($g-2$):** The most precisely tested prediction in the history of physics.
- **Delbrück Scattering:** The deflection of high-energy photons in the Coulomb field of heavy nuclei.
- **Light-by-Light Scattering:** Recently observed in lead-ion collisions at the **LHC (ATLAS/CMS)**.
- **Vacuum Birefringence:** Detected in the extreme magnetic environments of neutron stars.
- **Lamb Shift:** Confirming the interaction between the electron and vacuum fluctuation

Path Integrals in Quantum Mechanics

QED History & Path Integrals



- Alternative to Operator Methods.
- Generalization of Least Action
- The transition amplitude is a "sum" over all trajectories connecting x' to x

$$K(x, x'; t) \sim \sum_{\{x(\tau)\}} e^{iS[x(\tau)]}$$

QED History & Path Integrals



Key Publications in the Worldline Formalism

² From strings to particles Feynman 1950 "Mathematical formulation of the quantum theory of electromagnetic interaction"

Bern and Kosower 1991 "Efficient calculation of the one-loop QCD amplitudes"

M. J. Strassler 1992 "Field Theory without Feynman Diagrams: One-Loop Effective Actions"

QED History & Path Integrals



Advantages over the Feynman Diagrammatic Approach

- **Holistic Representation:**

- Treats scalar/spinor lines and loops as a single global object.
- Eliminates the need to decompose processes into individual propagators.

- **Permutation Invariance:**

- Does not require external photon legs to be ordered along the loop from the start.
- All $N!$ permutations are captured by a single expression.

- **Computational Efficiency:**

- **No Dirac Algebra:** Avoids tedious trace technology for closed-loop processes.
- **No Momentum Integrals:** At the one-loop level, the algebraic complexity is significantly reduced.

QED History & Path Integrals



Advantages over the Feynman Diagrammatic Approach.

- **Symmetry & Gauge Invariance:**

- Integration by parts (IBP) allows for the homogenization of integrands.
- Keeps gauge invariance manifest throughout the calculation, making underlying symmetries transparent.

One-loop N-point amplitude

One-loop effective action



Scalar case:

$$\Gamma_{\text{scalar}}[A] = \int_0^\infty \frac{dT}{T} e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x e^{-\int_0^T d\tau (\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu)}$$

Spinor case:

$$\Gamma_{\text{spin}}[A] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x e^{-\int_0^T d\tau (\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu)} \text{Spin}[x, A]$$

Where:

$$\text{Spin}[x, A] \rightarrow \int \mathcal{D}\psi(\tau) \exp \left[-\int_0^T d\tau \left(\frac{1}{2} \psi \cdot \dot{\psi} - ie\psi^\mu F_{\mu\nu} \psi^\nu \right) \right]$$

$$\psi(\tau_1)\psi(\tau_2) = -\psi(\tau_2)\psi(\tau_1)$$

$$\psi(T) = -\psi(0)$$

One-loop N-photon amplitude



$$\Gamma_{\text{scalar}}[A] = \int_0^\infty \frac{dT}{T} e^{-Tm^2} \int_{x(0)=x(T)} \mathcal{D}x e^{-\int_0^T d\tau (\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu)}$$

Polyakov 1987 (in "Gauge Fields and Strings")

Strassler 1992

In the presence of an N-photon background

$$A_\mu(x) = \sum_{i=1}^N \varepsilon_{i\mu} e^{ik_i \cdot x}$$

$$\Gamma_{\text{scal}}[\{k_i, \varepsilon_i\}] = (-ie)^N \int \frac{dT}{T} e^{-m^2 T} \int_{x(0)=x(T)} \mathcal{D}x e^{-\int_0^T d\tau \frac{1}{4}\dot{x}^2} V_{\text{scal}}[k_1, \varepsilon_1] V_{\text{scal}}[k_2, \varepsilon_2] \cdots V_{\text{scal}}[k_N, \varepsilon_N]$$

Photon Vertex Operator:

$$V_{\text{scal}}[k_i, \varepsilon_i] = \int_0^T d\tau_i \varepsilon_i \cdot \dot{x}(\tau_i) e^{ik_i \cdot x(\tau_i)}$$

One-loop N-photon amplitude



Master Formula:

$$\Gamma_{\text{scal}}(k_i, \varepsilon_i) = (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{-D/2} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \exp \left\{ \sum_{i,j=1}^N \left(\frac{1}{2} G_{Bij} k_i \cdot k_j - i \dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right) \right\} \Bigg|_{\varepsilon_1 \dots \varepsilon_N}$$

Where:

$$G(\tau_1, \tau_2) = |\tau_1 - \tau_2| - \frac{(\tau_1 - \tau_2)^2}{T}$$

$$\dot{G}(\tau_1, \tau_2) = \text{sgn}(\tau_1 - \tau_2) - \frac{2(\tau_1 - \tau_2)}{T}$$

$$\ddot{G}(\tau_1, \tau_2) = 2\delta(\tau_1 - \tau_2) - \frac{2}{T}$$

One-loop N-photon amplitude



Q-Representation

$$\Gamma_{\text{scal}}(k_i, \varepsilon_i) = (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{-D/2} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i \exp \left\{ \sum_{i,j=1}^N \left(\frac{1}{2} G_{Bij} k_i \cdot k_j - i \dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right) \right\} \Big|_{\varepsilon_1 \dots \varepsilon_N}$$

IBP:

$$\exp \left\{ \sum_{i,j=1}^N \left(\frac{1}{2} G_{Bij} k_i \cdot k_j - i \dot{G}_{Bij} \varepsilon_i \cdot k_j + \frac{1}{2} \ddot{G}_{Bij} \varepsilon_i \cdot \varepsilon_j \right) \right\} \Big|_{\varepsilon_1 \dots \varepsilon_N} \longrightarrow Q_N(\dot{G}_{Bij}) \exp \left\{ \frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j \right\}$$

• The polynomial Q:

- Homogeneous of degree N in the photon momenta
- Natural decomposition into cycles: It is defined as products of \dot{G}_{ij} 's with indices that form a closed cycle
- And certain left-overs called tails

One-loop N-photon amplitude



Q-Representation

$$\Gamma_{\text{scal}}(k_i, \varepsilon_i) = (-ie)^N \int_0^\infty \frac{dT}{T} (4\pi T)^{-D/2} e^{-m^2 T} \prod_{i=1}^N \int_0^T d\tau_i Q_N(\dot{G}_{Bij}) \exp \left\{ \frac{1}{2} \sum_{i,j=1}^N G_{Bij} k_i \cdot k_j \right\}$$

τ -cycle $\dot{G}(i_1 i_2 \dots i_N) = \dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \dots \dot{G}_{i_N i_1}$

"Lorentz-cycle" $Z_2(ij) \equiv \frac{1}{2} \text{tr}(f_i f_j) = (\varepsilon_i \cdot k_j)(\varepsilon_j \cdot k_i) - (\varepsilon_i \cdot \varepsilon_j)(k_i \cdot k_j)$

$$Z_n(i_1 i_2 \dots i_n) \equiv \text{tr} \left(\prod_{j=1}^n f_{i_j} \right) \quad (n \geq 3)$$

$$f_i^{\mu\nu} = k_i^\mu \varepsilon_i^\nu - \varepsilon_i^\mu k_i^\nu$$

Q-Representation Spinor result

Replacement Rule $\dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_k i_1} \rightarrow \dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_k i_1} - G_{F i_1 i_2} G_{F i_2 i_3} \cdots G_{F i_k i_1}$

Where $G_F(\tau_1, \tau_2) = \text{sgn}(\tau_1 - \tau_2)$

Integration Problem

$$\int_0^1 du_i \dot{G}_{i_1} \dot{G}_{i_2} \dot{G}_{i_3} = \int_0^1 du_i (\text{sgn}(u_i - u_1) - 2(u_i - u_1)) \cdots (\text{sgn}(u_i - u_3) - 2(u_i - u_3))$$

Six sectors
$$\begin{cases} 1 > u_1 > u_2 > u_3 > 0 \\ 1 > u_1 > u_3 > u_2 > 0 \\ \vdots \end{cases}$$

One-loop N-photon amplitude Integration Problem



Fixing

$$1 > u_1 > u_2 > u_3 > 0$$

$$\int_0^1 du_i \dot{G}_{i1} \dot{G}_{i2} \dot{G}_{i3} = \int_0^{u_3} du_i (-1 - 2u_i + 2u_1)(1 + 2u_i - 2u_2)(1 + 2u_i - 2u_3) + \int_{u_3}^{u_2} du_i (-1 - 2u_i + 2u_1)(-1 - 2u_i + 2u_2)(1 - 2u_i + 2u_3) \\ + \int_{u_2}^{u_1} du_i (-1 - 2u_i + 2u_1)(1 - 2u_i + 2u_2)(1 - 2u_i + 2u_3) + \int_{u_1}^1 du_i (1 - 2u_i + 2u_1)(1 - 2u_i + 2u_2)(1 - 2u_i + 2u_3)$$

After integration

$$\int_0^1 du_i \dot{G}_{i1} \dot{G}_{i2} \dot{G}_{i3} = -\frac{1}{6}(\dot{G}_{12} - \dot{G}_{23})(\dot{G}_{23} - \dot{G}_{31})(\dot{G}_{31} - \dot{G}_{12})$$

One-loop N-photon amplitude Integration Problem



- The inverse operator ∂_P^n is related with the Bernoulli polynomials $B_n(x)$

$$\langle u_i | \partial_P^{-n} | u_j \rangle = -\frac{1}{n!} B_n(|u_i - u_j|) \text{sign}^n(u_i - u_j)$$

$$\langle u_i | \partial_P^0 | u_j \rangle = \delta(u_i - u_j) - 1$$

- In this representation the Green function looks like

$$\frac{1}{2} G_{ij} = \langle u_i | \partial_P^{-2} | u_j \rangle + \frac{1}{12}$$

$$\frac{1}{2} \dot{G}_{ij} = \langle u_i | \partial_P^{-1} | u_j \rangle$$

$$\frac{1}{2} \ddot{G}_{ij} = \langle u_i | \partial_P^0 | u_j \rangle = \delta(u_i - u_j) - 1$$

One-loop N-photon amplitude

Integration Problem



- Integration properties

$$\int |u\rangle\langle u| du = 1$$

$$\int du_i du_j \langle u_i | \partial_P^{-n} | u_j \rangle = 0$$

$$\langle u | \partial_P^{-2n} | u \rangle = -\frac{B_{2n}}{(2n)!} = -\hat{B}_n$$

$$\langle u | \partial_P^{-(2n-1)} | u \rangle = 0$$

$$\langle u_i | \partial_P^{-n} | u_j \rangle = (-1)^n \langle u_j | \partial_P^{-n} | u_i \rangle$$

One-loop N-photon amplitude Integration Problem



Scalar cycle $\int_0^1 du_2 \int_0^1 du_3 \cdots \int_0^1 du_n \dot{G}_{12} \dot{G}_{23} \cdots \dot{G}_{n(n+1)}$

Two ingredients $\dot{G}_{ij} = 2\langle u_i | \partial_P^{-1} | u_j \rangle$ & $\int_0^1 du_j \langle u_i | \partial_P^{-1} | u_j \rangle \langle u_j | \partial_P^{-1} | u_k \rangle = \langle u_i | \partial_P^{-2} | u_k \rangle$

$$\int_0^1 du_2 \cdots \int_0^1 du_n \dot{G}_{12} \dot{G}_{23} \cdots \dot{G}_{n(n+1)} = 2^n \int_0^1 du_2 \cdots \int_0^1 du_n \langle u_1 | \partial_P^{-1} | u_2 \rangle \langle u_2 | \partial_P^{-1} | u_3 \rangle \cdots \langle u_n | \partial_P^{-1} | u_{n+1} \rangle$$

$$= 2^n \int_0^1 du_3 \cdots \int_0^1 du_n \langle u_1 | \partial_P^{-2} | u_3 \rangle \langle u_3 | \partial_P^{-1} | u_4 \rangle \cdots \langle u_n | \partial_P^{-1} | u_{n+1} \rangle$$

⋮

$$= 2^n \langle u_1 | \partial_P^{-n} | u_{n+1} \rangle = -\frac{2^n}{n!} B_n(|u_1 - u_{n+1}|) \text{sgn}^n(u_1 - u_{n+1})$$

One-loop N-photon amplitude



Low-energy limit

$$\omega_i \ll m, \quad i = 1, \dots, N.$$

$$V_{scal}[k_i, \varepsilon_i] = \int_0^T d\tau_i \varepsilon_i \cdot \dot{x}(\tau_i) e^{ik_i \cdot x(\tau_i)} \rightarrow V_{scal}^{(LE)}[k, \varepsilon] = \int_0^T d\tau \varepsilon \cdot \dot{x}(\tau) ik \cdot x(\tau).$$

Low-energy Photon vertex

$$V_{scal}^{(LE)}[f] = V_{scal}^{(LE)}[k, \varepsilon] - \frac{i}{2} \int_0^T d\tau \frac{d}{d\tau} (\varepsilon \cdot x(\tau) k \cdot x(\tau)) = \frac{i}{2} \int_0^T d\tau x(\tau) \cdot f \cdot \dot{x}(\tau)$$

$$f^{\mu\nu} \equiv k^\mu \varepsilon^\nu - \varepsilon^\mu k^\nu$$

One-loop N-photon amplitude



The scalar QED N-photon amplitudes in terms of Wick contractions of vertex operators

$$\Gamma_{\text{scal}}[k_1, \varepsilon_1; \dots; k_N, \varepsilon_N] = (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} (4\pi T)^{-\frac{D}{2}} \left\langle V_{\text{scal},1}^A \cdots V_{\text{scal},N}^A \right\rangle$$

Low-energy limit

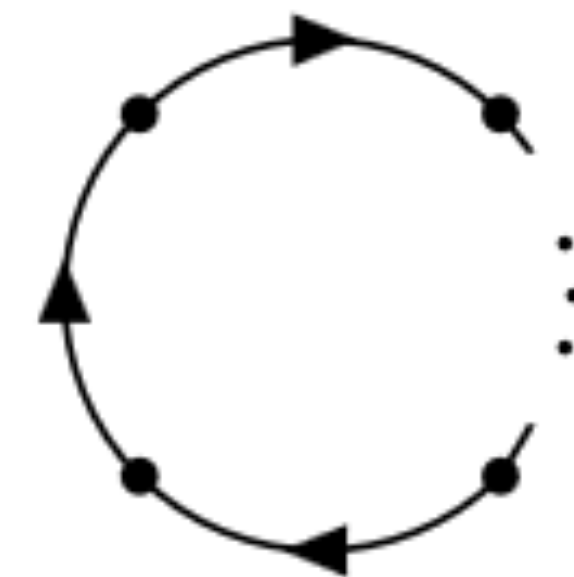
$$\left\langle V_{\text{scal}}^{\gamma(\text{LE})}[f_1] \cdots V_{\text{scal}}^{\gamma(\text{LE})}[f_n] \right\rangle = i^n T^n \exp \left\{ \sum_{r=1}^{\infty} \frac{b_{2r}}{4r} \text{tr}(f_{\text{tot}}^{2r}) \right\} \Big|_{f_1 \cdots f_n}$$

Where

$$f_{\text{tot}} \equiv \sum_{i=1}^N f_i$$

Identified with the one-loop N-point

$$b_n \equiv \int_0^T d\tau_{i_1} \cdots \int_0^T d\tau_{i_n} \dot{G}_{i_1 i_2} \dot{G}_{i_2 i_3} \cdots \dot{G}_{i_n i_1} = -2^n \frac{B_n}{n!}$$



One-loop N-photon amplitude

Low-energy limit



$$\Gamma_{\text{scal}}^{(\text{LE})}[\{k_i, \varepsilon_i\}] = (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} (4\pi T)^{-D/2} i^N T^N \exp \left\{ \sum_{r=1}^{\infty} \frac{b_{2r}}{4r} \text{tr}(f_{\text{tot}}^{2r}) \right\} \Bigg|_{f_1 \dots f_n}$$

$$= \frac{e^N \Gamma(N-2)}{(4\pi)^2 m^{2N-4}} \exp \left\{ \sum_{r=1}^{\infty} \frac{b_{2r}}{4r} \text{tr}(f_{\text{tot}}^{2r}) \right\} \Bigg|_{f_1 \dots f_n}$$

4-Photons

$$\exp \left(\sum_{m=1}^{\infty} \frac{b_{2m}}{4m} \text{tr}(f_{\text{tot}}^{2m}) \right) \Bigg|_{f_1 \dots f_n} = \frac{b_4}{8} \text{tr}(f_{\text{tot}}^4) + \frac{1}{2!} \left(\frac{b_2}{4} \text{tr}(f_{\text{tot}}^2) \right)^2$$

$$\Gamma_{\text{scal}}^{(\text{LE})}[\{k_i, \varepsilon_i\}] = = \frac{e^4 \Gamma(2)}{(4\pi)^2 m^4} \left(\frac{b_4}{8} \text{tr}(f_{\text{tot}}^4) + \frac{1}{2!} \frac{b_2^2}{4^2} \text{tr}(f_{\text{tot}}^2)^2 \right)$$

One-loop N-photon amplitude

Low-energy limit



$$\Gamma_{\text{scal}}^{(\text{LE})}[\{k_i, \varepsilon_i\}] = (-ie)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} (4\pi T)^{-D/2} i^N T^N \exp \left\{ \sum_{r=1}^{\infty} \frac{b_{2r}}{4r} \text{tr}(f_{\text{tot}}^{2r}) \right\} \Big|_{f_1 \dots f_n}$$

$$= \frac{e^N \Gamma(N-2)}{(4\pi)^2 m^{2N-4}} \exp \left\{ \sum_{r=1}^{\infty} \frac{b_{2r}}{4r} \text{tr}(f_{\text{tot}}^{2r}) \right\} \Big|_{f_1 \dots f_n}$$

All helicities '+'

$$f_{\text{tot}}^2 = -\chi_+ \quad \text{Whith}$$

$$\chi_+ \equiv \frac{1}{2} \sum_{1 \leq i < j \leq N} [k_i k_j]^2$$

$$\text{tr}(f_{\text{tot}}^{2m}) = 4(-1)^m (\chi_+)^m$$

$$\exp \left\{ \sum_{m=1}^{\infty} \frac{b_{2m}}{4m} \text{tr}(f_{\text{tot}}^{2m}) \right\} \Big|_{f_1 \dots f_N} = \exp \left\{ \sum_{m=1}^{\infty} (-1)^m \frac{b_{2m}}{m} \chi_+^m \right\} \Big|_{\text{all different}}$$

$$-2 \ln \frac{\sin \sqrt{\chi_+}}{\sqrt{\chi_+}}$$

One-loop N-photon amplitude

Low-energy limit



All helicities '+'

$$\Gamma_{\text{scal}}^{(\text{LE})}(k_1, \varepsilon_1; \dots; k_N, \varepsilon_N) = -\frac{m^4}{(4\pi)^2} \left(\frac{2ie}{m^2}\right)^N \frac{\mathcal{B}_N}{N(N-2)} \chi_N^+$$

Where

$$\chi_N^+ \equiv \chi_+^{\frac{N}{2}} \Big|_{\text{all different}} = \frac{\left(\frac{N}{2}\right)!}{2^{\frac{N}{2}}} \left\{ [k_1 k_2]^2 [k_3 k_4]^2 \cdots [k_{N-1} k_N]^2 + \text{all distinct permutations} \right\}$$

For example N=4

$$\chi_4^+ = \frac{1}{2} \left\{ [k_1 k_2]^2 [k_3 k_4]^2 + [k_1 k_3]^2 [k_2 k_4]^2 + [k_1 k_4]^2 [k_2 k_3]^2 \right\}$$

$$\chi_4^+ = \frac{1}{2}(s^2 + t^2 + u^2)$$

Conclusions

The Power of the Worldline Formalism

- The worldline approach has proven to be highly efficient for calculating N-photon amplitudes, significantly reducing the complexity compared to traditional Feynman diagrams.

Persistent Mathematical Challenges

- Despite its elegance, significant challenges remain, particularly in handling higher-loop corrections and extending these techniques to more complex non-Abelian gauge theories.

Conclusions

No "Single Superior Path"

- Since its inception, Quantum Mechanics has provided us with diverse frameworks (from Matrix Mechanics to Path Integrals). History shows that there is no "absolute superior" method; rather, different formalisms offer unique insights.

The Importance of Community

- Meetings like this are essential. Discussing diverse methodologies allows us to cross-pollinate ideas and find the most robust tools for the next generation of problems in Quantum Field Theory.



Thanks!